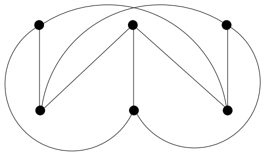
Planar Graphs (1)

Topological graph theory, broadly conceived, is the study of graph layouts. Contemporary applications include circuit layouts on silicon chips. Wire crossings cause problems in layouts, so we ask which circuits have layouts without crossings.

Example. Can three (enemies) A, B, and C build roads to three utilities so that the roads do not cross? In other words, can $K_{3,3}$ be drawn without edge crossings?



Planar Graphs (2)

Arguments about drawings of graphs in the plane are based on the fact that every closed curve in the plane separates the plane into two regions (the inside and the outside). Some of the arguments used here are somewhat intuitive, since full details in topology are difficult.

Theorem 6.1.2. K_5 and $K_{3,3}$ cannot be drawn without crossings.

Proof: Consider a drawing of K_5 or $K_{3,3}$ in the plane, and let C be a spanning cycle drawn as a closed curve. Chords of C must be drawn inside or outside this curve (a *chord* of a cycle is an edge not in C whose endpoints lie in C). If the endpoints on C of two chords occur in alternating order, they conflict, and one must be drawn inside C and the other outside C.

Planar Graphs (3)

Proof: (cont.) A 6-cycle in $K_{3,3}$ has three pairwise conflicting chords, so it is not possible to complete the embedding. A 5-cycle in K_5 has five chords, out of which no more than two are pairwise nonconflicting; again it is not possible to complete the embedding. □

Obviously, the complete graph K_n with $n \geq 6$ have K_5 as a subgraph and cannot be planar. One may then search for the **crossing number**, the minimum number of crossings in a drawing in the plane. For K_n with $n \geq 5$, the crossing numbers are 1, 3, 9, 18, 36, 60, $\leq 100, \ldots$; see <URL:http://www.research.att.com/~njas/sequences/index.html>.

Planar Graphs (4)

drawing A function f defined on $V(G) \cup E(G)$ that assigns each vertex v a point $f(v)$ in the plane and assigns each edge with endpoints u and v a (polygonal) curve from $f(u)$ to $f(v)$. It is required that $f(u) \neq f(v)$ for distinct $u, v \in V(G)$.

crossing A point in $f(e) \cap f(e')$ that is not a common endpoint.

planar embedding A drawing without crossings.

planar A graph is planar if it has a planar embedding.

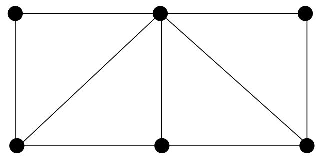
plane graph A particular planar embedding of a planar graph.

face Maximal regions of the plane that contain no point used in the embedding (the unbounded face is called the *outer face*).

Outerplanar Graphs

A graph is **outerplanar** if it has an embedding with every vertex on the boundary of the unbounded face. An **outerplane graph** is such an embedding of an outerplanar graph.

Example. The graphs K_4 and $K_{2,3}$ are planar but not outerplanar. An example of an outerplane graph is as follows:



Euler's Formula (1)

Euler's Formula is the basic counting tool relating vertices, edges, and faces in planar graphs.

Theorem 6.1.21. If a connected plane graph G has exactly n vertices, e edges, and f faces, then $n - e + f = 2$.

Proof: We use induction on the number of vertices, n.
Basis step: $n = 1$. The only vertices are loops. If $e = 0$, then $f = 1$ and the formula holds. Each added loop passes through a face and cuts it into two faces. This augments the edge count and the face count each by 1, and therefore the formula holds for any number of edges.

Euler's Formula (2)

Proof: (cont.) Induction step: $n > 1$. Since G is connected, we can find an edge that is not a loop. When we contract such an edge, we obtain a plane graph G' with $n' = n - 1$, $e' = e - 1$, and $f' = f$. By the induction hypothesis,

$$n - e + f = n' + 1 - (e' + 1) + f' = n' - e' + f' = 2,$$

which completes the proof. □

Euler's Formula generalizes for plane graphs with k components: $n - e + f = k + 1$.

Euler's Formula (3)

Theorem 6.1.23. If G is a simple planar graph with at least three vertices, then $e(G) \leq 3n(G) - 6$. If also G is triangle-free, then $e(G) \leq 2n(G) - 4$.

Proof: It suffices to consider connected graphs; otherwise we could add edges. We count the edges two ways, directly and as boundaries of faces (the number of edges that form the boundary of face i is denoted by f_i):

$$2e = \sum_i f_i \geq 3f.$$

Substituting into Euler's formula yields $e \leq 3n - 6$. When G is triangle-free, we get $2e \geq 4f$ and $e \leq 2n - 4$. □

Euler's Formula (4)

Theorem 6.1.27. A graph embeds in the plane iff it embeds on a sphere.

Proof: Given an embedding on a sphere, we can puncture the sphere inside a face and project the embedding onto a plane tangent to the opposite point (the punctured face on the sphere becomes the outer face in the plane). The process is reversible. □

Theorem 6.1.27 may be used to count the facets of polyhedra. Moreover, it may be utilized to show that there are only five regular polyhedra (the Platonic solids): tetrahedron, cube, octahedron, dodecahedron, icosahedron [Wes, Application 6.1.28].

Kuratowski's Theorem (1)

A *subdivision* of a graph is a graph obtained from it by replacing edges by pairwise internally-disjoint paths.

Theorem 6.2.1. If a graph G has a subgraph that is a subdivision of K_5 or $K_{3,3}$, then G is nonplanar.

Proof: Follows from Theorem 6.1.2, by observing that every subgraph of a planar graph is planar, and subdivision does not affect planarity. □

Kuratowski proved that these necessary conditions are also sufficient (TONCAS); this result is **Kuratowski's Theorem**.

Kuratowski's Theorem (2)

Kuratowski subgraph A subgraph that is a subdivision of K_5 or $K_{3,3}$.

minimal nonplanar graph A nonplanar graph such that every proper subgraph is planar.

Kuratowski's theorem can be proved in two steps:

1. A minimal nonplanar graph with no Kuratowski subgraph must be 3-connected [Wes, Lemma 6.2.7].

2. Every 3-connected graph with no Kuratowski subgraph is planar [Wes, Theorem 6.2.11].

The Four-Color Theorem (1)

The problem of determining the maximal chromatic number of planar graphs is perhaps the most famous problem in graph theory.

Theorem. Every planar graph is 6-colorable.

Proof: Every simple n-vertex planar graph has at most $3n-6$ edges (Theorem 6.1.23). Such a graph must have at least one edge with degree at most 5 (otherwise there would be at least $6n/2 = 3n$ edges). Such a vertex can always be colored, so we may delete the vertex. The same argument applies to the resulting graph. □

The Four-Color Theorem (2)

The proof of the following strengthening of the previous theorem can be found in [Wes].

Theorem 6.3.1. Every planar graph is 5-colorable.

The **Four-Color Theorem** was finally proved by Appel, Haken, and Koch in the 1970s using sophisticated arguments and a lot of CPU time. The proof has later been refined, but still depends on computers (a contemporary computer only needs minutes to prove this result). This bound is strict: K_4 is planar and needs four colors.

Theorem 6.3.6. Every planar graph is 4-colorable.

Edge-Colorings (1)

Proper vertex coloring: a color class is an independent set.
Proper edge-coloring: a color class is a matching.

An edge-coloring of G may also be viewed as a vertex coloring of the line graph of G.

k-edge-coloring A labelling $f : E(G) \to S$, where $|S| = k$.
proper Each color class is a matching.
k-edge-colorable A graph that has a proper k-edge-coloring.
edge-chromatic number, chromatic index The least k such that a graph G is k-edge-colorable; denoted by $\chi'(G)$.

Edge Colorings (2)

Note: In contrast to $\chi(G)$, $\chi'(G)$ is affected by multiple edges. A graph with a loop has no proper edge-coloring.

Obviously $\chi'(G) \geq \Delta(G)$. An easy upper bound is $2\Delta(G) - 1$: since no edge has a common endpoint with more than $2(\Delta(G) - 1)$ edges, we can color the edges using $2\Delta(G) - 1$ colors, one by one in any order.

Theorem 7.1.7 If G is bipartite, then $\chi'(G) = \Delta(G)$.

For a regular graph, a proper edge-coloring with $\Delta(G)$ colors is equivalent to a 1-factorization.

Edge Colorings (3)

It is easy to find simple graphs for which $\chi'(G) > \Delta(G)$. For example, $\chi'(C_{2m+1}) = 3 > \Delta(C_{2m+1}) = 2$. Can the difference between $\chi'(G)$ and $\Delta(G)$ be greater than 1? No!

Theorem 7.1.10. If G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.

For simple graphs, we have only two possibilities for $\chi'(G)$:

Class 1 $\chi'(G) = \Delta(G)$.
Class 2 $\chi'(G) = \Delta(G) + 1$.

Hamiltonian Cycles

Finding a Hamiltonian cycle is a special instance of the traveling salesman problem (TSP).

Hamiltonian cycle A spanning cycle.
Hamiltonian graph A graph with a Hamiltonian cycle.

No easily testable characterization is known for Hamiltonian graphs; some necessary and sufficient conditions are studied here. Loops and multiple edges are obviously irrelevant, so we restrict our attention to simple graphs.

We first look at *necessary* conditions.

Hamiltonian Cycles; Necessary Conditions (1)

Example. A spanning cycle in a bipartite graph visits the two partite sets alternately, so the partite sets must have the same size. For example, $K_{m,n}$ is Hamiltonian only if $m = n$.

Denote the number of components of a graph H by $c(H)$.

Theorem 7.2.3. If G has a Hamiltonian cycle, then for each nonempty set $S \subseteq V$, $c(G - S) \leq |S|$.

Proof: When leaving a component of $G - S$, a Hamiltonian cycle can go only to S, and the arrivals in S must use distinct vertices of S. See figure on [Wes, p. 287]. □

Hamiltonian Cycles; Necessary Conditions (2)

Example 1. Every Hamiltonian graph is 2-connected. Namely, by Theorem 7.2.3, the graph $G - v$ has at most one component.

Example 2. The first graph in [Wes, Example 7.2.5] is bipartite with partite sets of equal sizes. However, it fails the necessary condition of Theorem 7.2.3. *For what choice of S?* The second graph in [Wes, Example 7.2.5] satisfies Theorem 7.2.3 but has no spanning cycle.

Example 3. The Petersen graph is another example of a non-Hamiltonian graph that satisfies the condition in Theorem 7.2.3. The only 2-factor of the Petersen graph is $2C_5$.

Hamiltonian Cycles; Sufficient Conditions (1)

We now turn to *sufficient* conditions. One would assume that with many edges in the graph, a Hamiltonian cycle would be forced. We focus on the minimum degree of a graph.

Example. There are graphs with minimum degree $\lfloor (n-1)/2 \rfloor$ that are not Hamiltonian: take, for example, complete graphs of orders $\lfloor (n-1)/2 \rfloor$ and $\lceil (n+1)/2 \rceil$ sharing a vertex (this graph is not even 2-connected), or, for odd n, take $K_{(n-1)/2,(n+1)/2}$. These graphs are illustrated in [Wes, Example 7.2.7].

Hamiltonian Cycles; Sufficient Conditions (2)

Theorem 7.2.8. If G is a simple graph with at least three vertices and $\delta(G) \geq n(G)/2$, then G is Hamiltonian.

Proof: Adding edges cannot reduce the minimum degree. We assume that there are non-Hamiltonian graphs with minimum degree at least $n(G)/2$ and consider a maximal—in the sense that adding any edge makes the graph Hamiltonian—such graph G.

When $u \not\leftrightarrow v$, the maximality of G implies that there is a spanning path $v_1, v_2, \ldots, v_n$ with endpoints $u = v_1$ and $v = v_n$. If a neighbor of u follows a neighbor of v on the path, $u \leftrightarrow v_{i+1}$ and $v \leftrightarrow v_i$, then $(u, v_{i+1}, v_{i+2}, \ldots, v, v_i, v_{i-1}, \ldots, v_2)$ is a spanning cycle (see picture on [Wes, p. 289]).

Hamiltonian Cycles; Sufficient Conditions (3)

Proof: (cont.) To prove that a neighbor of u on the path follows a neighbor of v, we prove that the sets defined by $S = \{i : u \leftrightarrow v_{i+1}\}$ and $T = \{i : v \leftrightarrow v_i\}$ are overlapping. We get

$$|S \cup T| + |S \cap T| = |S| + |T| = d(u) + d(v) \geq n.$$

Neither S nor T contains the index n. Thus $|S \cup T| \leq n - 1$, and hence $|S \cap T| \geq 1$. We have thereby established a desired contradiction. $\square$

Theorem 7.2.8 and the preceding example solves an extremal problem: The maximum value of $\delta(G)$ among non-Hamiltonian n-vertex graphs is $\lfloor (n-1)/2 \rfloor$.

Hamiltonian Cycles; Sufficient Conditions (4)

The hypotheses and the proof of Theorem 7.2.8 can be somewhat weakened:

- For the minimum degree, we only require that $d(u) + d(v) \geq n$ whenever $u \not\leftrightarrow v$.
- We just need the fact that there exists a spanning path of length $n - 1$.

Theorem 7.2.9 Let G be a simple graph. If $u, v \in V(G)$ are distinct and nonadjacent with $d(u) + d(v) \geq n$, then G is Hamiltonian iff $G + uv$ is Hamiltonian.

Hamiltonian Cycles; Sufficient Conditions (5)

The **(Hamiltonian) closure** of a graph G, denoted by $C(G)$, is the graph with vertex set $V(G)$ obtained from G by iteratively adding edges joining pairs of nonadjacent vertices whose degree sum is at least n, until no such pair remains. The closure is well-defined [Wes, Lemma 7.2.12].

Example. See figure after [Wes, Definition 7.2.10].

Theorem 7.2.11. A simple n-vertex graph is Hamiltonian iff its closure is Hamiltonian.

Hamiltonian Cycles; Sufficient Conditions (6)

We show one more sufficient conditions related to the degree of vertices.

Theorem 7.2.13 Let G be a simple graph with vertex degrees $d_1 \le d_2 \le \cdots \le d_n$, where $n \ge 3$. If $i < n/2$ implies that $d_i > i$ or $d_{n-i} \ge n - i$ (Chvátal's condition), then G is Hamiltonian.

Hamiltonian Paths

A **Hamiltonian path** is a spanning path.

Every graph with a Hamiltonian cycle has a Hamiltonian path, but the converse is not true (shown, for example, by P_n).

Theorem 7.2.16. A graph G has a Hamiltonian path iff H has a Hamiltonian cycle, where $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G) \cup \{uv : u \in V(G)\}$.