Connectivity (1)

A good communication network is hard to disrupt. We want the graph (or digraph) to be connected even when some vertices or edges fail. In the subsequent discussion, graphs are assumed to be loopless.

separating set, vertex cut A set $S \subseteq V(G)$ such that $G - S$ has more than one component.
connectivity The minimum size of a separating set (or $n(G) - 1$ if no such set exists); denoted by $\kappa(G)$.
k-connected A graph with $\kappa(G) \geq k$.

Connectivity (2)

Some (characterization) results:

- ▷ $\kappa(G) = n(G) - 1$ iff G is a complete graph ($\kappa(G) \leq n(G) - 2$ for all other graphs).
- ▷ $\kappa(G) = 1$ iff G is connected and has a cut-vertex.
- ▷ $\kappa(G) = 0$ iff G is disconnected (but $\kappa(K_1) = 0$).
- ▷ $\kappa(G) \leq \delta(G)$, since the neighbors of an arbitrary vertex form a separating set.

Connectivity of Hypercubes (1)

We have $\kappa(Q_n) \leq \delta(Q_n) = n$. Therefore, to prove that $\kappa(Q_n) = n$, it suffices to prove that $\kappa(Q_n) \geq n$. We use induction on n.
Basis step: $n \in \{0, 1\}$. Obvious.
Induction step: $n \geq 2$. We consider Q_n as two copies of Q_{n-1}, Q and Q', with a (perfect) matching between their corresponding vertices. Let S be a vertex cut in Q_n. If $Q - S$ is connected and $Q' - S$ is connected, then $Q_n - S$ is also connected unless S contains at least one endpoint of every matched pair. This requires $|S| \geq 2^{n-1}$, but $2^{n-1} \geq n$ for $n \geq 2$.

Connectivity of Hypercubes (2)

(Cont.) Hence we may assume that $Q - S$ is disconnected, which means that S has at least $n - 1$ vertices in Q, by the induction hypothesis. If S contains no vertices of Q', then $Q' - S$ is connected and all vertices of $Q - S$ have neighbors in $Q' - S$, so $Q_n - S$ is connected. Hence S must also contain a vertex of Q'. This yields $|S| \geq n$, as desired.

Note: Since $\kappa(G) = k$ requires $\delta(G) \geq k$, it also requires at least $\lceil kn(G)/2 \rceil$ edges. This bound is indeed best possible; *Harary graphs* are k-connected graphs of this size.

Harary Graphs (1)

Place n ($> k$) vertices around a circle. We get three cases for forming $H_{k,n}$:

1. If k is even, make each vertex adjacent to the nearest $k/2$ vertices in each direction around the circle.
2. If k is odd and n is even, make each vertex adjacent to the nearest $(k-1)/2$ vertices in each direction around the circle and to the diametrically opposite vertex.
3. If k and n are both odd, construct $H_{k,n}$ from $H_{k-1,n}$ by adding the edges $i \leftrightarrow i + (n-1)/2$ for $0 \leq i \leq (n-1)/2$ (with the vertices indexed $0, 1, \ldots, n-1$ around the circle).

Harary Graphs (2)

Example. See the graphs in [Wes, Example 4.1.4].

Theorem 4.1.5. $\kappa(H_{k,n}) = k$, and hence the minimum number of edges in a k-connected graph on n vertices is $\lceil kn/2 \rceil$.

Proof Techniques for Connectivity

Direct proof of $\kappa(G) \geq k$: Consider a vertex cut and prove that $|S| \geq k$, or consider a set S with fewer than k vertices and prove that $G - S$ is connected.

Indirect proof of $\kappa(G) \geq k$: Assume a vertex cut of size less than k and find a contradiction.

Proving $\kappa(G) = k$: Prove that $\kappa(G) \geq k$ (see above) and $\kappa(G) \leq k$ by presenting a vertex cut of size k (usually easy).

Edge-Connectivity

Instead of deleting vertices (failing nodes), we may delete edges (failing links).

disconnecting set A set $F \subseteq E(G)$ such that $G - F$ has more than one component.

edge cut An edge set of the form $[S, V(G) - S]$, where $[S, T]$ denotes the set of edges with endpoints in both S and T.

bond A minimal edge cut.

edge-connectivity The minimum size of a disconnecting set; denoted by $\kappa'(G)$.

k-edge-connected A graph with $\kappa'(G) \geq k$.

Example. See figure in [Wes, Definition 4.1.7].

Connectivity vs. Edge-Connectivity (1)

For a given vertex u of minimum degree $\delta(G)$, the vertices adjacent to u form a separating set and the edges incident to u form a disconnecting set, both of size $\delta(G)$, which is then an upper bound on $\kappa(G)$ and $\kappa'(G)$.

Deleting one endpoint of each edge in a disconnecting set F deletes every edge of F, suggesting that $\kappa(G) \leq \kappa'(G)$ (but we have to take care that we indeed have more than one components in the end).

Theorem 4.1.9. If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Connectivity vs. Edge-Connectivity (2)

Example. The graph in [Wes, Example 4.1.10] has $\kappa(G) < \kappa'(G) < \delta(G)$ ($\kappa = 1$, $\kappa' = 2$, and $\delta = 3$).

Theorem 4.1.11. If G is a 3-regular graph, then $\kappa(G) = \kappa'(G)$.

Proof: Since $\kappa(G) \leq \kappa'(G)$ by Theorem 4.1.9, we need only provide a disconnecting set of size $|S|$, where S is a minimum vertex cut. Let H_1 and H_2 be the two components of $G - S$. Since S is a minimum vertex cut, each $v \in S$ has a neighbor in H_1 and a neighbor in H_2. As v has degree 3, v cannot have two neighbors in both H_1 and H_2. The edge in the disconnecting set is then the edge from v to the member of $\{H_1, H_2\}$ where v has only one neighbor.

Connectivity vs. Edge-Connectivity (3)

Proof: (cont.) The aforementioned edges break all paths from H_1 to H_2 except in the case shown on [Wes, p. 154], where a path can enter S via v_1 and leave via v_2. In this case we delete the edge to H_1 for both v_1 and v_2 to break all paths from H_1 to H_2 through $\{v_1, v_2\}$. □

block A maximal connected subgraph (that is, possibly the whole graph) that has no cut-vertex.

k-Connected Graphs (1)

The concept of k-connectedness is related to the property of having several alternative paths between vertices. When $k = 1$, this connection is obvious: G is 1-connected iff each pair of vertices is connected by a path.

Two paths are said to be **internally disjoint** if they have no common internal vertex.

Theorem 4.2.2. A graph G having at least three vertices is 2-connected iff for each pair $u, v \in V(G)$ there exist internally disjoint paths from u to v.

k-Connected Graphs (2)

A characterization of 2-edge-connected graphs is given in [Wes, Theorem 4.2.10].

Connectivity of digraphs is analogous to that of undirected graphs; now the question is whether subgraphs obtained after deleting vertices or edges are *strongly* connected or not.

Further results on k-connectedness are discussed in the algorithm part of the course.

Line Graphs

From a given graph, we may form a new graph by interchanging the roles of vertices and edges.

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G and $ef \in E(L(G))$ when $e = uv$ and $f = vw$ are edges in G (for *line digraphs*, the head of e must be the tail of f).

Example. See the graphs in [Wes, Definition 4.2.18].

Vertex Coloring (1)

Vertex colorings and some related concepts were already defined in the introductory part (but are included here for completeness).

k-coloring A labelling $f : V(G) \to S$, where $|S| = k$ (often $S = \{1, 2, \ldots, k\} =: [k]$). The labels are *colors* and the vertices of one color form a *color class*.
proper A coloring where adjacent vertices have different colors.
k-colorable A graph that has a proper k-coloring.
chromatic number The least k such that a graph G is k-colorable; denoted by $\chi(G)$.

Note: vertex coloring = proper coloring

Vertex Coloring (2)

We may assume that we have loopless graphs. Namely, graphs with loops are uncolorable and multiple edges are irrelevant.

Recall that a graph is k-colorable iff $V(G)$ is the union of k independent sets, that is, G is k-partite.

Example. The graphs C_5 and the Petersen graph are not 2-colorable, since they are not bipartite. Since they are 3-colorable, as shown in [Wes, Example 5.1.3], they have chromatic number 3.

Vertex Coloring (3)

k-chromatic A graph with $\chi(G) = k$.
optimal coloring A proper k-coloring of a k-chromatic graph.
color-critical A graph G with the property that $\chi(H) < \chi(G)$ for every proper subgraph of G. If $\chi(G) = k$, the graph may also be called k-critical.

Example. Properly coloring a graph needs at least two colors iff the graphs has at least one edge. So K_2 is the only 2-critical graph (and K_1 is the only 1-critical graph). Since 2-colorable = bipartite, the characterization of bipartite graphs (no odd cycle!) implies that the 3-critical graphs are C_{2n+1}, $n \geq 1$. (No good characterizations of 4-critical graphs or test for 3-colorability is known.)

Clique Number

The **clique number** of a graph G, written $\omega(G)$, is the maximum size of a clique (set of pairwise adjacent vertices) in G. Recall that $\omega(G) = \alpha(\overline{G})$.

Theorem 5.1.7. $\chi(G) \geq \omega(G)$ and $\chi(G) \geq n(G)/\alpha(G)$.

Proof: The first bound holds because all the vertices of a clique must be assigned different colors in a proper coloring. The second bound holds because every color class is an independent set and thus has at most $\alpha(G)$ vertices. $\square$

Example. $\chi(G)$ may exceed $\omega(G)$, as shown by $G = C_{2r+1}$, $r \geq 2$. Then $\chi(G) = 3$ (from earlier example) and $\omega(G) = 2$.

Graph Products

In general, a graph product of two graphs G and H is a new graph whose vertex set is $V(G) \times V(H)$ and where, for any two vertices (g, h) and (g', h') in the product, the adjacency of those two vertices is determined entirely by the adjacency (or equality, or nonadjacency) of g and g' and that of h and h' in the original graphs.

There are $3 \cdot 3 - 1 = 8$ cases to be decided (three possibilities for each, with the case where both are equal eliminated), and thus there are $2^8 = 256$ different types of graph products that can be defined. Clearly, only a few of these are of more general interest, such as the cartesian product.

Cartesian Product (1)

The **cartesian product** of G and H, $G \square H$ (other notations occur in the literature), is a graph product with adjacency defined by

$$(g, h) \leftrightarrow (g', h') \text{ iff } (g = g' \text{ and } h \leftrightarrow h') \text{ or } (g \leftrightarrow g' \text{ and } h = h').$$

Example. The hypercube can be viewed as a cartesian product: $Q_n = K_2 \square K_2 \square \cdots \square K_2$ (n times). The $m \times n$ *grid graph* is the graph $P_m \square P_n$. The graph $C_3 \square C_4$ is depicted in [Wes, Example 5.1.10].

Cartesian Product (2)

The cartesian product allows us to compute chromatic numbers by computing independence numbers.

Theorem. A graph G is m-colorable iff the cartesian product $G \square K_m$ has an independent set of size $n(G)$ [Wes, Exercise 5.1.31].

Upper Bounds for Chromatic Numbers

Theorem 5.1.13. $\chi(G) \leq \Delta(G) + 1$.

Proof: By using $\Delta(G) + 1$ colors, we can color the vertices of G, one by one in any order, since there are always at least $(\Delta(G) + 1) - \Delta(G) = 1$ colors that do not occur among the adjacent vertices. $\square$

We can strengthen this result.

Theorem 5.1.22. If G is connected and neither a complete graph nor an odd cycle, then $\chi(G) \leq \Delta(G)$.

Example. Theorem 5.1.22 proves that the Petersen graph is 3-colorable.

Other Types of Colorings

So far, we have only considered vertex colorings. *Edge colorings* are considered later in this course. There are also other types of colorings of vertices than those presented here.

For *generalized colorings*, we have other requirements on the vertices of the color classes. For *list colorings*, we restrict the colors allowed to be used on each vertex.

T. R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley, New York, 1994.

Structure of k-chromatic Graphs

By Theorem 5.1.7, $\chi(G) \geq \omega(G)$ for all G. If this equality holds for a graph G and all its induced subgraphs, we say that G is **perfect**. We shall next see how *bad* this bound can be.

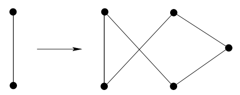
The average values of $\omega(G)$, $\alpha(G)$, and $\chi(G)$ over all graphs with vertex set $[n]$ are close to $2\lg n$, $2\lg n$, and $n/(2\lg n)$, respectively. This indicates that $\omega(G)$ is a bad lower bound on $\chi(G)$ and that the other lower bound in Theorem 5.1.7, $\chi(G) \geq n(G)/\alpha(G)$, is better.

It turns out that there are triangle-free graphs ($\omega(G) = 2$) that have arbitrarily large chromatic number.

Mycielski's Construction (1)

From a simple graph G, *Mycielski's construction* produces a simple graph G' containing G. Starting from G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, add vertices $U = \{u_1, u_2, \ldots, u_n\}$ and a vertex w. Add edges to make u_i adjacent to all of $N_G(v_i)$, and let $N_{G'}(w) = U$.

Example 1.



Example 2. Starting from C_5, we get the *Grötzsch graph*; see [Wes, Example 5.2.2].

Mycielski's Construction (2)

Theorem 5.2.3. From a k-chromatic triangle-free graph G, Mycielski's construction produces a $(k+1)$-chromatic triangle-free graph G'.

Proof: Since U is an independent set in G', the other vertices of a triangle containing u_i must belong to $V(G)$ and be neighbors of v_i. This would lead to a triangle in $V(G)$ (replacing u_i by v_i), which does not exist. Hence G' is triangle-free.

A proper k coloring f of G extends to a proper $(k+1)$-coloring of G' by setting $f(u_i) = f(v_i)$ and $f(w) = k+1$; hence $\chi(G') \le \chi(G) + 1$. We prove that $\chi(G) < \chi(G')$ by considering any proper coloring of G' and obtaining a proper coloring of G using fewer colors.

Mycielski's Construction (3)

Proof: (cont.) Let g be a proper k-coloring of G'. W.l.o.g., we assume that $g(w) = k$, which restricts g to $\{1, 2, \ldots, k-1\}$ on U. Let A be the set of vertices in G on which g uses color k; we shall next change these colors to obtain a proper $(k-1)$-coloring of G.

For each $v_i \in A$, we change the color to $g(u_i)$. Since $N_{G'}(u_i) \cap G = N_{G'}(v_i) \cap G$, we get a proper coloring after the change unless the color of some of these neighbors have also been changed. However, this is not possible, since $g(v_i) = k$ implies that for all $v_j \in N_G(v_i)$, we have $g(v_j) \ne k$ and therefore $v_j \notin A$. $\square$

Turán's Theorem (1)

What are the smallest and largest k-chromatic graphs with n vertices.

Theorem 5.2.5. Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges. Equality holds for K_k plus $n-k$ isolated vertices.

A **complete multipartite graph** is a simple graph G whose vertices can be partitioned into sets so that $u \leftrightarrow v$ iff u and v belong to different sets of the partition.

The **Turán graph** $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ in size by at most 1 (that is, they have size $\lceil n/r \rceil$ or $\lfloor n/r \rfloor$).

Turán's Theorem (2)

Theorem 5.2.8. Among simple r-colorable (that is, r-partite) graphs with n vertices, the Turán graph is the unique graph with the most edges.

We shall give a somewhat stronger result, named after Turán who published it in 1941; *Turán's theorem* is viewed as the origin of extremal graph theory.

Theorem 5.2.9. Among the n-vertex simple graphs with no $(r+1)$-clique, $T_{n,r}$ is the (unique [Wes, Exercise 5.2.21]) graph with the maximum number of edges.