Trees

acyclic graph A graph with no cycle.
forest An acyclic graph.
tree A connected acyclic graph.
leaf A vertex of degree 1.
spanning subgraph A subgraph of G with vertex set $V(G)$.
spanning tree A spanning subgraph that is a tree.

Example. See the graph in [Wes, Definition 2.1.1].

Properties of Trees (1)

Lemma 2.1.3. Every tree with at least two vertices has at least two leaves.

Proof: A connected graph with at least two vertices has an edge. Consider any maximal path in the tree. Since the tree is acyclic, the endpoints of the path must be vertices of degree 1, that is, leaves. □

Deleting a leaf from an n-vertex tree obviously produces a tree with $n-1$ vertices.

Properties of Trees (2)

Trees have many characterizations.

Theorem 2.1.4. For an n-vertex graph ($n \geq 1$), the following are equivalent and characterize the trees with n vertices.

A) G is connected and has no cycles.
B) G is connected and has $n-1$ edges.
C) G has $n-1$ edges and no cycles.
D) For $u, v \in V(G)$, there is exactly one path from u to v.

We shall now prove some parts of this theorem; see [Wes, Theorem 2.1.4.] for the rest of the cases.

Properties of Trees (3)

Proof: A $\Rightarrow$ {B,C}. Use induction on n. The basis step, $n = 1$, is trivial. For $n > 1$, given an acyclic connected graph G, Lemma 2.1.3 provides a leaf v and states that $G' = G - v$ is acyclic and connected. Applying the induction hypothesis to G' yields $e(G') = n - 2$. Since only one edge is incident to v, we have $e(G) = n - 1$.

D $\Rightarrow$ A. If there is a path from u to v for all $u, v \in V(G)$, then G is connected. If G has a cycle C, then G has (at least) two paths from u to v for $u, v \in V(C)$; hence G is acyclic. □

Properties of Trees (4)

Corollary 2.1.5. a) Every edge of a tree is a cut-edge.
b) Adding one edge to a tree forms exactly one cycle.
c) Every connected graph contains a spanning tree.

Proof: a) Deleting an edge leaves $n-2$ edges, so we cannot have a tree (Theorem 2.1.4B), and since there are no cycles, the new graph must be disconnected (Theorem 2.1.4A).
b) A tree has a unique path linking each pair of vertices (Theorem 2.1.4D), so joining two vertices by an edge creates exactly one cycle.
c) Delete edges from cycles in a connected graph until there are $n-1$ edges left. □

Properties of Trees (5)

Theorem 2.1.8. If T is a tree with k edges and G is a simple graph with $\delta(G) \geq k$, then T is a subgraph of G.

Proof: Use induction on k. Basis step: $k = 0$. Obvious.
Induction step: $k > 0$. Assume that the claim holds for trees with fewer than k edges. Since $k > 0$, there is a leaf v in T by Lemma 2.1.3; let u be its neighbor. Consider the smaller tree $T' = T - v$. By the induction hypothesis, G contains T' as a subgraph, since $\delta(G) \geq k > k-1$. Let x be the vertex of G corresponding to u. Because T' has $k-1$ vertices other than u and $d_G(x) \geq k$, x has a neighbor y in G that is not in this copy of T'. The edge xy expands T' into T, with y playing the role of v. □

Properties of Trees (6)

The inequality of Theorem 2.1.8 is sharp: the graph K_k has minimum degree $k-1$, but it contains no tree with k edges.

Theorem 2.1.8 implies that every n-vertex simple graph with more than $n(k-1)$ edges has T as a subgraph [Wes, Exercise 2.1.34]. Erdős and Sós conjectured the stronger statement that $e(G) > n(k-1)/2$ forces T as a subgraph (which has been proved for graphs without 4-cycles).

Distance

If G has a path from u to v, then the **distance** from u to v, written $d_G(u,v)$ or $d(u,v)$, is the least length of such a path. If no path exists, $d(u,v) = \infty$.

diameter diam $G := \max_{u,v \in V(G)} d(u,v)$.
eccentricity The eccentricity of a vertex u is
$\epsilon(u) := \max_{v \in V(G)} d(u,v)$.
radius rad $G := \min_{u \in V(G)} \epsilon(u)$.

The diameter can also be viewed as the maximum of the vertex eccentricities.

Example: Distance

In the first three examples, every vertex has the same eccentricity, and diam $G =$ rad G.

- ▷ The Petersen graph has diameter 2, since nonadjacent verteces have a common neighbor.
- ▷ The hypercube Q_n has diameter n, since it takes n steps to change all n coordinates.
- ▷ The cycle C_n has diameter $\lfloor n/2 \rfloor$.
- ▷ For $n \geq 3$, the n-vertex tree of least diameter is the star $K_{1,n-1}$, with diameter 2 and radius 1. The one with the largest diameter is the path P_n, with diameter $n-1$ and radius $\lceil (n-1)/2 \rceil$. In general, the diameter of a tree is the length of its longest path.

Diameter of Complement

Theorem 2.1.11. If G is a simple graph, then diam $G \geq 3 \Rightarrow$ diam $\overline{G} \leq 3$.

Proof: Since diam $G \geq 3$, there exist nonadjacent $u, v \in V(G)$ with no common neighbor. Hence every $x \in V(G) - \{u, v\}$ has at least one of u and v as a nonneighbor, and therefore as a neighbor in $\overline{G}$. Since $uv \in V(\overline{G})$, for every x, y there is a path in $\overline{G}$ from x to y with length at most $1 + 1 + 1 = 3$. □.

Center

The **center** of a graph is the subgraph induced by the vertices of minimum eccentricity. The center of a graph is the full graph iff the radius and diameter are equal.

Theorem 2.1.13. The center of a tree is a vertex or an edge.

In a communication network, large diameter may be acceptable if *most* pairs can communicate via short parths. This leads to the study of *average* distance instead of *maximum* distance.

Wiener Index

Since the average distance is the sum of distances divided by $\binom{n}{2}$, we may simply study the **Wiener index**

$$D(G) := \sum_{u,v \in V(G)} d_G(u, v).$$

Theorem 2.1.13. Among trees with n vertices, the Wiener index $D(T)$ is minimized by stars and maximized by paths, both uniquely.

Enumerating trees

There are $2^{\binom{n}{2}}$ graphs with n vertices. Out of these, n^{n-2} are trees (Cayley's formula). This result can also be interpreted as the number of spanning trees in a complete graph of order n.

A **contraction** of edge $e = uv$ is the replacement of u and v with a single vertex whose incident edges are the edges that were incident to u or v, except for e. The resulting graph is denoted by $G \cdot e$. Let $\tau(G)$ denote the number of spanning trees of G.

Theorem 2.2.8. If $e \in E(G)$ is not a loop, then $\tau(G) = \tau(G - e) + \tau(G \cdot e)$.

Matching

We briefly introduce the concept of matching, which will be considered more thoroughly in the algorithm part.

matching A set of non-loop edges with no shared endpoints.
The endpoints (vertices) of the edges of a matching are said to be *saturated*; the other vertices are *unsaturated*.

perfect matching A matching saturating every vertex.

Example. Yet another example of the difference between *maximum* objects and *maximal* objects are given in [Wes, Example 3.1.5] for matchings.

Vertex Cover

A **vertex cover** of a graph G is a set $Q \subseteq V(G)$ that contains at least one endpoint of every edge. We say that the vertices in Q *cover* $E(G)$. (We want to minimize the size of vertex covers.)

Since no vertex can cover two edges of a matching, the size of every vertex cover is at least the size of every matching. Therefore, obtaining a matching and vertex cover of the same size *proves* that each is optimal.

Example. See [Wes, Example 3.1.15].

Independence Number

The **independence number** of a graph is the maximum size of an independent set of vertices.

The independence number of a bipartite graph may exceed the sizes of the partite sets. An example where this happens is given in [Wes, Example 3.1.18].

Edge Cover

An **edge cover** of a graph G is a set $L \subseteq E(G)$ such that every vertex of G is incident to some edge of L. We say that the edges in Q *cover* $V(G)$. (We want to minimize the size of edge covers.)

- ▷ Only graphs without isolated vertices have edge covers.
- ▷ There are at least $n(G)/2$ edges in an edge cover; a perfect matching forms an edge cover attaining this bound.

Notations

maximum size of independent set	$\alpha(G)$
maximum size of matching	$\alpha'(G)$
minimum size of vertex cover	$\beta(G)$
minimum size of edge cover	$\beta'(G)$

Example. For the 5-cycle, we have $\alpha(C_5) = 2$, $\alpha'(C_5) = 2$, $\beta(C_5) = 3$, and $\beta'(C_5) = 3$.

Graph Parameters (1)

Theorem 3.1.21. In a graph G, $S \subseteq V(G)$ is an independent set iff $V(G) - S$ is a vertex cover.

Proof: If S is an independent set, then every edge is incident to at least one vertex of $V(G) - S$. Conversely, if $V(G) - S$ covers all the edges, then no edge can have both endpoints in S, so S is indeed an independent set. □

Corollary. $\alpha(G) + \beta(G) = n$.

Graph Parameters (2)

Theorem 3.1.16. If G is a bipartite graph, then $\alpha'(G) = \beta(G)$.

Proof: [R. Rizzi, *J. Graph Theory* **33** (2000), 138–139] Assume that counterexamples to the theorem exist, and pick a minimal (with respect to deleting vertices and edges) such graph G. Obviously, G is connected and is neither an even cycle nor a path, so G has a vertex u of degree at least 3. Let v be a vertex adjacent to u. Assume that $\alpha'(G - v) < \alpha'(G)$. Then, by minimality, $G - v$ has a vertex cover W with $|W| = \alpha'(G - v) \leq \beta(G) - 2$. Since $W \cup \{v\}$ is a vertex cover of G, we get $\beta(G) \leq \alpha'(G - v) + 1$, a contradiction.

Graph Parameters (3)

Proof: (cont.) If $\alpha'(G-v) = \alpha'(G)$, then there exists a maximum matching M of G that does not saturate v. Let f be an edge of $G - M$ such that u and f are incident (such an edge exists because u has degree at least 3). By minimality, $\alpha'(G-f) = \beta(G-f)$; such a minimum vertex cover, W', covers all edges of a maximal matching of the same size. Since it also covers uv, and no edge of M has v as an endpoint, it follows that W' contains u. Therefore, W' covers G and $\beta(G-f) = \beta(G) = \alpha'(G)$, a contradiction. □

This result is called *König's matching theorem* or the *König-Egerváry theorem*.

Graph Parameters (3)

Theorem 3.1.22. If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.

Proof: 1) From a maximum matching M, we construct an edge cover of size $n(G) - |M|$: Add to M one edge incident to each unsaturated vertex. Since each edge in M takes care of two vertices, the size of the edge cover is indeed $n(G) - |M|$.
2) From a minimum edge cover L, we construct a matching of size $n(G) - |L|$: If both endpoints of an edge e belong to edges in L other than e, then $e \notin L$ (since L is a minimum cover). It follows that each component formed by edges of L is a star. Let k be the number of such components. Obviously, $|L| = n(G) - k$, and a matching of size k is obtained by choosing one edge from each component. □

Graph Parameters (4)

Corollary 3.1.24. If G is a bipartite graph with no isolated vertices, then $\alpha(G) = \beta'(G)$.

Proof: By the Corollary of Theorem 3.1.21 and by Theorem 3.1.22, we get $\alpha(G) + \beta(G) = \alpha'(G) + \beta'(G)$. Applying Theorem 3.1.16 ($\alpha'(G) = \beta(G)$) completes the proof. □

Dominating Sets (1)

In vertex covers and edge covers, vertices cover edges and vice versa. One may also consider the case where vertices cover adjacent vertices.

In a graph G, a set $S \subseteq V(G)$ is a **dominating set** if every vertex not in S has a neighbor in S. The **domination number** $\gamma(G)$ is the minimum size of a dominating set in G.

Example. See [Wes, Example 3.1.27].

Dominating Sets (2)

A dominating set S in G is called *connected* if S induces a connected subgraph, and *independent* if S is an independent set in G.

Example 1. A connected dominating set can be used as a virtual backbone for routing in wireless ad hoc networks.

Example 2. A dominating set in an n-cube can be used as a gambling system for the football pools.

Factors

factor A spanning subgraph (called a *k-factor* if the subgraph is k-regular).

factorization A decomposition of a graph into factors (called a *k-factorization* if the factors are k-factors).

Note: A 1-factor and a perfect matching is almost the same thing, the only difference being that a 1-factor is a subgraph whereas a perfect matching is the set of edges in such a subgraph.