Matroids (beyond the greedy algorithm)

Matroids have many connections to combinatorial optimization beyond the greedy algorithm (see Volume B of [Sch]). We sketch three examples that illustrate a small part of these connections.

1. Matroid intersection
- Maximum bipartite matching is a special case of optimization in the intersection of two matroids.

2. Matroid union
- The union of two matroids is a matroid; this enables easy proofs of certain packing and covering results and gives efficient algorithms for obtaining such packings.
- As an example, we prove the Nash-Williams-Tutte theorem on packing spanning trees in a graph.

3. The rank polynomial
- The rank polynomial of a matroid is an object with connections to many branches of mathematics ranging from coding theory to knot theory (see [God, Wel2]).
- We sketch a connection between the rank polynomial of $M(G)$ and a seemingly unrelated polynomial, the chromatic polynomial of G (see [Big, Tut, Wel2]).

Sources

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Matroid intersection

Let M_1 and M_2 be hereditary systems on E. The **intersection** $M_1 \cap M_2$ is the hereditary system on E defined by $\mathbf{I}_{M_1 \cap M_2} = \mathbf{I}_{M_1} \cap \mathbf{I}_{M_2}$. The sets in $\mathbf{I}_{M_1 \cap M_2}$ are the **common** independent sets of M_1 and M_2.

In general, $M_1 \cap M_2$ is not a matroid even if M_1 and M_2 are matroids.

Example. Let G be a bipartite graph with bipartition X, Y. Assume the edges in G are oriented from X to Y. The **head (tail) partition matroid** M_{head} (M_{tail}) is the partition matroid induced by the partition of $E(G)$ into edges with common head (tail) vertex. The common independent sets of M_{head} and M_{tail} are precisely the matchings in G.

The matroid intersection theorem

The following theorem is due to Edmonds (1970).

Theorem A.47 (Matroid Intersection Theorem) *For matroids M_1, M_2 on E, the size of a largest common independent set satisfies*

$$\max\{|I| \ : \ I \in \mathbf{I}_{M_1} \cap \mathbf{I}_{M_2}\} = \min\{r_{M_1}(X) + r_{M_2}(\bar{X}) \ : \ X \subseteq E\}.$$

Proof: See [Wes, p. 367]. $\square$

Exercise: Prove the König-Egerváry theorem using the matroid intersection theorem on $M_{\text{head}} \cap M_{\text{tail}}$.

Matroid union

The **union** $M_1 \cup \cdots \cup M_k$ of hereditary systems $M_1, \ldots, M_k$ on E is the hereditary system on E defined by

$$\mathbf{I}_{M_1 \cup \cdots \cup M_k} = \big\{I_1 \cup \cdots \cup I_k \ : \ I_i \in \mathbf{I}_{M_i} \text{ for all } i \in \{1, \ldots, k\}\big\}.$$

The **direct sum** $M_1 \oplus \cdots \oplus M_k$ of hereditary systems $M_1, \ldots, M_k$ on pairwise disjoint sets $E_1, \ldots, E_k$ is the hereditary system on $E_1 \cup \cdots \cup E_k$ defined by

$$\mathbf{I}_{M_1 \oplus \cdots \oplus M_k} = \big\{I_1 \cup \cdots \cup I_k \ : \ I_i \in \mathbf{I}_{M_i} \text{ for all } i \in \{1, \ldots, k\}\big\}.$$

Exercise: Prove that $M_1 \oplus \cdots \oplus M_k$ is a matroid if $M_1, \ldots, M_k$ are matroids.

The matroid union theorem

The following theorem is due to Nash-Williams (1967) and Edmonds (1968).

Theorem A.48 (Matroid Union Theorem) *Let $M_1, \ldots, M_k$ be matroids on E. Then, the union $M = M_1 \cup \cdots \cup M_k$ is a matroid on E with rank function*

$$r_M(X) = \min\left\{|X - Y| + \sum_{i=1}^{k} r_{M_i}(Y) \ : \ Y \subseteq X\right\}.$$

Proof: See [Wes, p. 370–371]. $\square$

Applications of the matroid union theorem

The following theorem is due to Edmonds (1965).

Corollary A.4 (Matroid Packing Theorem) *Given a matroid M on E, the maximum number of pairwise disjoint bases equals*

$$\min\left\{\left\lfloor\frac{|E| - |X|}{r(E) - r(X)}\right\rfloor \ : \ X \subseteq E, \ r(X) < r(E)\right\}.$$

Proof: Let $k \geq 0$ and put $M' = M_1 \cup \ldots \cup M_k$, where $M_i := M$ for all $i \in \{1, \ldots, k\}$. By the matroid union theorem, there are k pairwise disjoint bases in M if and only if $r_{M'}(E) \geq k r_M(E)$, or equivalently, $|E| - |X| + k r_M(X) \geq k r_M(E)$ for all $X \subseteq E$. $\square$

The following theorem is due to Nash-Williams (1961) and Tutte (1961).

Corollary A.5 *A graph G has k pairwise edge-disjoint spanning trees if and only if for every partition P of $V(G)$, there are at least $k(|P|-1)$ edges with endpoints in different sets of P.*

Proof: We may assume that G is connected. By the matroid packing theorem, G has k edge-disjoint spanning trees if and only if the cycle matroid $M(G)$ satisfies for all $X \subseteq E(G)$

$$|E(G)| - |X| \geq k\big(r(E(G)) - r(X)\big).$$

Since $r(\sigma(X)) = r(X)$ and $X \subseteq \sigma(X)$, the above condition holds if and only if it holds for all closed subsets of $E(G)$.

Proof: (continued) A subset $X \subseteq E(G)$ is closed in $M(G)$ if and only if $X = \bigcup_{i=1}^{p} E(G[V_i])$ for some partition $P = \{V_1, \ldots, V_p\}$ of $V(G)$.

Without loss of generality we may assume that each $G[V_i]$ is connected since splitting V_i into the component vertex sets of $G[V_i]$ does not change X or the number of edges between different sets of P (but increases $|P| = p$).

Let $X \subseteq E(G)$ be any closed set and let P be the corresponding partition of $V(G)$ into sets inducing connected subgraphs. We have $r(X) = n(G) - p$ and, since G is connected, $r(E(G)) = n(G) - 1$. The claim follows because $k\big(r(E(G)) - r(X)\big) = k(p-1)$ and $|E(G)| - |X|$ is the number of edges between different sets of P. □

Algorithms for matroid intersection/union

- Polynomial time algorithms exist for many matroid problems (see Volume B of [Sch]), including the following:
  1. for matroids M_1, M_2 on E and $w : E \to \mathbb{R}$, find a common independent set of the maximum total weight;
  2. for a matroid M on E, find the minimum number of independent sets that cover E;
  3. for a matroid M on E, find the maximum number of pairwise disjoint bases in E.
- Polynomial time means in this context polynomial in $|E|$ together with the assumption that the independence of a set $X \subseteq E$ can be tested in time polynomial in $|E|$.

The rank polynomial

Let M be a matroid on E. The **rank polynomial** of M is the polynomial

$$R(M; x, y) := \sum_{A \subseteq E} x^{r(E)-r(A)} y^{r^*(E)-r^*(A)}.$$

In particular, $R(M; x, y)$ enumerates the subsets of E by rank and corank.

Theorem A.49 *Let M be a matroid on E and let $e \in E$. Then,*

$$R(M; x, y) = \begin{cases} (1+y)R(M-e; x, y) & \text{if } e \text{ is a loop;} \\ (1+x)R(M \cdot e; x, y) & \text{if } e \text{ is a coloop;} \\ R(M-e; x, y) + R(M \cdot e; x, y) & \text{otherwise.} \end{cases}$$

Let P be the Petersen graph. The coefficients of $R(M(P);x,y)$ are

$$
R(M(P);x,y) = \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ x^4 \\ x^5 \\ x^6 \\ x^7 \\ x^8 \\ x^9 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 1 & y & y^2 & y^3 & y^4 & y^5 & y^6 \end{array} \\
\left(\begin{array}{ccccccc}
2000 & 2172 & 1230 & 445 & 105 & 15 & 1 \\
4680 & 2765 & 816 & 135 & 10 & 0 & 0 \\
5805 & 1725 & 240 & 15 & 0 & 0 & 0 \\
4875 & 630 & 30 & 0 & 0 & 0 & 0 \\
2991 & 130 & 0 & 0 & 0 & 0 & 0 \\
1365 & 12 & 0 & 0 & 0 & 0 & 0 \\
455 & 0 & 0 & 0 & 0 & 0 & 0 \\
105 & 0 & 0 & 0 & 0 & 0 & 0 \\
15 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

The chromatic polynomial

Let G be a graph and let $P(G,t)$ be the number of proper colorings of $V(G)$ with t colors. For example,

$$P(K_n,t) = t(t-1)(t-2)\cdots(t-n+1), \qquad P(\bar{K}_n,t) = t^n.$$

Perhaps somewhat surprisingly, the function $P(G,t)$ is always a polynomial. Indeed,

$$P(G,t) = \begin{cases} 0 & \text{if } G \text{ contains a loop; and} \\ P(G-e,t) - P(G\cdot e,t) & \text{otherwise (for any } e \in E(G)\text{).} \end{cases}$$

The polynomial $P(G,t)$ is the **chromatic polynomial** of G.

Theorem A.50 *Let $\mathcal{M}$ be a family of matroids that is closed under restriction and contraction, and let F be a function on $\mathcal{M}$ that associates a real polynomial in x,y to each $M \in \mathcal{M}$.*

If there exist $a,b \in \mathbb{R}$ such that for all $M \in \mathcal{M}$,

$$F(M) = \begin{cases} 1 & \textit{if } M \textit{ is empty;} \\ a(1+y)F(M-e) & \textit{if } e \textit{ is a loop;} \\ b(1+x)F(M\cdot e) & \textit{if } e \textit{ is a coloop;} \\ aF(M-e)+bF(M\cdot e) & \textit{otherwise.} \end{cases}$$

then, for all $M \in \mathcal{M}$,

$$F(M) = a^{|E|-r(E)} b^{r(E)} R(M;x,y),$$

where M is defined on E.

Proof: By induction on $|E|$. $\square$

Theorem A.51 *Let G be a graph with n vertices and c components. Then,*

$$P(G,t) = (-1)^{n-c} t^c R(M(G);-t,-1).$$

Proof: Theorem A.50 is not directly applicable because $P(G,t)$ is not constant for $\bar{K}_n$. Instead, we apply Theorem A.50 to $\hat{P}(G,t) := t^{-c}P(G,t)$, where c is the number of components in G.

If $G = \bar{K}_n$, we have $c = n$ and $\hat{P}(G,t) = t^{n-c} = 1$. If G contains a loop, $\hat{P}(G,t) = 0$. If $e \in E(G)$ is a coloop (i.e. a cut-edge), $P(G-e,t) = tP(G\cdot e,t)$, which implies

$$\hat{P}(G,t) = t^{-c}(t-1)P(G\cdot e,t) = (t-1)\hat{P}(G\cdot e,t).$$

For other $e \in E(G)$, we have

$$\hat{P}(G,t) = \hat{P}(G-e,t) - \hat{P}(G\cdot e,t).$$

Proof: (continued) We may view $\hat{P}(G,t)$ as a function of $M(G)$ and t because $M(G-e) = M(G) - e$ and $M(G \cdot e) = M(G) \cdot e$. Thus,

$$\hat{P}(M(G),t) = \begin{cases} 1 & \text{if } M(G) \text{ is empty;} \\ 0 & \text{if } M(G) \text{ has a loop;} \\ (t-1)\hat{P}(M(G)\cdot e, t) & \text{if } e \text{ is a coloop;} \\ \hat{P}(M(G)-e,t) - \hat{P}(M(G)\cdot e, t) & \text{otherwise.} \end{cases}$$

To apply Theorem A.50 to $\hat{P}$, we require

$$a(1+y) = 0, \quad b(1+x) = t-1, \quad a = 1, \quad b = -1.$$

Therefore, $x = -t$ and $y = -1$. Theorem A.50 now gives $\hat{P}(G,t) = (-1)^{n-c}R(M(G); -t, -1)$; multiplying by t^c gives the theorem. $\square$

6. NP-complete problems and graph isomorphism

- All is not lost even when we are faced with a problem for which no polynomial time algorithm is known, such as any **NP**-complete problem or the graph isomorphism problem.
- As with any computational problem, the first task is to obtain recent reference material (such as a survey article) on the problem.
- Then, the task is to analyze the problem at hand in detail:
  1. What kinds of instances need to be solved?
  2. What is required of the solutions?

Types of instances: special cases

- Many problems that are hard **in general** have **special cases** that can be solved in polynomial time. For example, all of the following special cases have polynomial time algorithms.
  1. Locate a maximum clique in a planar graph.
  2. Locate a maximum clique in a graph of bounded maximum degree.
  3. Locating a vertex cover of the minimum size in a bipartite graph.
  4. Test graphs of bounded maximum degree for isomorphism.

Types of instances: small size/easy majority

- For small instances, even a worst-case exponential time algorithm will typically get the job done.
- For some problems, the majority of instances can be quickly solved with an appropriate algorithm (although instances requiring exponential time do exist).
- For example, the *nauty* graph isomorphism package available at `http://cs.anu.edu.au/~bdm/nauty/` can relatively quickly test most graphs with up to a thousand vertices for isomorphism.

Types of solutions: approximability

- For many hard optimization problems there exist polynomial time algorithms that are guaranteed to find an **approximate solution** that is close to optimal.
- For example, it is **NP**-hard to find a vertex cover of the minimum size $\beta(G)$ for an arbitrary graph G.
- We can find an approximately optimal vertex cover by computing (in polynomial time) a maximum matching M in G. The set of vertices in M is a vertex cover of size $2\alpha'(G)$.
- Since $\alpha'(G) \leq \beta(G) \leq 2\alpha'(G)$, this vertex cover is never more than twice the size of a minimum vertex cover.

Types of solutions: inapproximability

- There also exist problems for which essentially no good approximation algorithm can exist unless $\mathbf{P} = \mathbf{NP}$.
- Examples of such problems are the traveling salesperson problem (TSP) and the maximum clique problem.
- Again, appropriate special case instances may be approximable.
- For example, TSP has relatively good approximation algorithms if the intercity distances satisfy the triangle equality.

Types of algorithms

1. Exact algorithms
  - backtrack search, branch and bound

2. Approximation algorithms
  - linear programming (primal-dual schema), greedy approximation

3. Randomized algorithms
  - abundance of witnesses, random sampling, LP relaxation and random rounding

4. Heuristic algorithms
  - hill-climbing, simulated annealing, tabu search, genetic algorithms

More on algorithmics for hard problems

- A recent textbook on algorithmics for hard problems is [Hro].
- For exact algorithms and computing isomorphism, see [Kre]. (T-79.5202 Combinatorial Algorithms)
- For more on heuristic search, see [Aar, Kre]. (T-79.5204 Combinatorial Models and Stochastic Algorithms)
- For approximation algorithms, see [Aus, Vaz], and for randomized algorithms, see [Mot].
- For computational complexity theory, see [Gar, Pap]. (T-79.5103 Computational Complexity Theory)

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