

6. Locally Dependent Events

- The "Lovász Local Lemma" (Erdős & Lovász 1975).
- Consider a probabilistic proof for the existence of a "good" configuration (sample point) $\omega \in \Omega$ that has none of the "bad" properties A_1, \dots, A_n ; i.e. none of the events A_1, \dots, A_n occur in ω , i.e. $\omega \in \bigcap_i \bar{A}_i$.
- The aim is thus to prove that $\Pr(\bigcap_i \bar{A}_i) > 0$.
- If each of the bad events has prob. $\Pr(A_i) = p_i < 1$, and the events are mutually independent, then

$$\Pr(\bigcap_i \bar{A}_i) = \prod_i (1 - p_i) > 0,$$

and we are done.

- However, if the events A_i have dependencies, we can only argue that

$$\Pr(\bigcap_i \bar{A}_i) = \Pr(\overline{\bigcup_i A_i}) = 1 - \Pr(\bigcup_i A_i) \geq 1 - \sum_i \Pr(A_i).$$

Also this is OK, if there are not too many events A_i , and/or the probabilities $p_i = \Pr(A_i)$ are small.

(And in the general case this is the best possible bound, viz. when the events are disjoint, $A_i \cap A_j = \emptyset$ for $i \neq j$.)

- Nevertheless one can get better bounds when the dependencies among events are local. (E.g. in random graphs, events pertaining to disjoint vertex sets are [often] independent, and a given vertex set intersects only a bounded number of others.)

- A dependency graph for a set of events A_1, \dots, A_n is a graph G on vertex set $[n] = \{1, \dots, n\}$ s.t. event A_i is mutually independent of all the events A_j s.t. $j \neq i$ in G (and also all their Boolean combinations).

- Theorem 6.1 (LLL, Erdős & Lovász 1975). Let G be a dependency graph for events A_1, \dots, A_n . Assume that $\deg(G_i) \leq d$ and $\Pr(A_i) \leq p$ for all $i = 1, \dots, n$. If $4pd \leq 1$, then $\Pr(\bigcap \bar{A}_i) > 0$.

Proof. We first prove by induction on n that for any n events A_1, \dots, A_n (re-indexed as necessary):

$$\Pr(A_1 \mid \bar{A}_2 \dots \bar{A}_n) \leq 2p. \quad \left(\begin{array}{l} \bar{A}_2 \dots \bar{A}_n \text{ shorthand} \\ \text{for } \bar{A}_2 \cap \dots \cap \bar{A}_n \end{array} \right)$$

For $n=1$, $\Pr(A_1) \leq p \leq 2p$ by assumption.

For $n > 1$, let for a given event A_1 , events A_2, \dots, A_k be all its neighbours according to G . (Note $k-1 \leq d$.)

Then, applying the decomposition $\Pr(A \mid BC) = \Pr(ABC) / \Pr(BC)$:

$$(*) \quad \Pr(A_1 \mid \bar{A}_2 \dots \bar{A}_n) = \frac{\Pr(A_1 \bar{A}_2 \dots \bar{A}_k \mid \bar{A}_{k+1} \dots \bar{A}_n)}{\Pr(\bar{A}_2 \dots \bar{A}_k \mid \bar{A}_{k+1} \dots \bar{A}_n)}$$

The numerator of (*) can be upperbounded by independence:

$$\begin{aligned} \Pr(A_1 \bar{A}_2 \dots \bar{A}_k \mid \bar{A}_{k+1} \dots \bar{A}_n) &\leq \Pr(A_1 \mid \bar{A}_{k+1} \dots \bar{A}_n) \\ &= \Pr(A_1) \leq p \end{aligned}$$

and denominator lowerbounded by induction:

$$\begin{aligned} \Pr(\bar{A}_2 \dots \bar{A}_k \mid \bar{A}_{k+1} \dots \bar{A}_n) &= 1 - \Pr\left(\bigcup_{i=2}^k A_i \mid \bar{A}_{k+1} \dots \bar{A}_n\right) \\ &\geq 1 - \sum_{i=2}^k \Pr(A_i \mid \bar{A}_{k+1} \dots \bar{A}_n) \\ &\geq 1 - (k-1) \cdot 2p \geq \frac{1}{2}. \end{aligned}$$

Note:
 $k-1 \leq d$
 $2pd \leq 1/2$

Thus,

$$\Pr(A_1 | \bar{A}_2 \dots \bar{A}_m) \leq \frac{p}{1/2} = 2p,$$

completing the induction.

To obtain the claim of the theorem, note that

$$\begin{aligned} \Pr(\bar{A}_1 \dots \bar{A}_n) &= \prod_{i=1}^n \Pr(\bar{A}_i | \bar{A}_1 \dots \bar{A}_{i-1}) \\ &\geq (1-2p)^n \\ &> 0. \quad \square \end{aligned}$$

• Theorem 6.2 (Generalised LL, Spencer 1977). Let G be a dependency graph for events A_1, \dots, A_n . Assume that there exist real numbers x_1, \dots, x_n , $0 \leq x_i < 1$, s.t. for all i :

$$\Pr(A_i) \leq x_i \cdot \prod_{j \sim i} (1-x_j).$$

Then

$$\Pr(\bigcap_i \bar{A}_i) \geq \prod_{i=1}^n (1-x_i).$$

Proof. As above, with the induction hypothesis

$$\Pr(A_1 | \bar{A}_2 \dots \bar{A}_m) \leq x_1,$$

and lowerbounding the denominator in (*) as

$$\begin{aligned} \Pr(\bar{A}_2 \dots \bar{A}_k | \bar{A}_{k+1} \dots \bar{A}_m) &= \prod_{j=2}^k \Pr(\bar{A}_j | \bar{A}_{j+1} \dots \bar{A}_m) \\ &\geq \prod_{j=2}^k (1-x_j) \\ &= \prod_{j=1}^k (1-x_j). \quad \square \end{aligned}$$

• Example 1. Ramsey numbers.

Recall: $R(k) > n$ if there is a two-colouring of the edges of K_n that induces no monochromatic k -cliques.

Consider a random two-colouring of K_n , and associate to each k -set of vertices $S \subseteq [n]$ the event $A_S \sim$ " S is monochromatic". Define a dependency graph for these events by

$$S \sim T \text{ if } |S \cap T| \geq 2.$$

Applying the LLL with $\Pr(A_S) = 2 \cdot \left(\frac{1}{2}\right)^{\binom{k}{2}} = 2^{1 - \binom{k}{2}}$ and

$$d = \binom{k}{2} \binom{n}{k-2}$$

yields:

Theorem 6.3 If $4 \cdot 2^{1 - \binom{k}{2}} \cdot \binom{k}{2} \binom{n}{k-2} < 1$, then $R(k) > n$. \square

Working out the asymptotics, this results in:

$$R(k) > \frac{\sqrt{2}}{e} \cdot k 2^{k/2} (1 + o(1)).$$

This is a factor of $\sqrt{2}$ improvement over the bound obtained by the alteration method (Thm 3.1)

• Example 2. Hypergraph 2-colourings.

A hypergraph with n vertices is any family \mathcal{F} of subsets of $[n] = \{1, \dots, n\}$.

A hypergraph \mathcal{F} is k -uniform if every "hyperedge" $S \in \mathcal{F}$ has exactly k elements. (Thus, a standard graph is a 2-uniform hypergraph.)

A hypergraph \mathcal{F} is r -regular if every vertex is contained in exactly r hyperedges.

A hypergraph \mathcal{F} is 2-colourable (or "has property B", where B stands either for [Felix] Bernstein or "bipartite"), if the vertices can be 2-coloured so that no $S \in \mathcal{F}$ is monochromatic.

• Theorem 6.4 (Erdős 1963). Every k -uniform hypergraph with fewer than 2^{k-1} edges is 2-colourable.

Proof. Let \mathcal{F} be a k -uniform hypergraph with $|\mathcal{F}| < 2^{k-1}$. Consider a random 2-colouring of the vertices of \mathcal{F} . Then for each $S \in \mathcal{F}$, $\Pr(S \text{ is mono-}\chi) = 2 \cdot 2^{-k}$, and so:

$$\begin{aligned} \Pr(\exists S \in \mathcal{F} : S \text{ mono-}\chi) &\leq \sum_{S \in \mathcal{F}} \Pr(S \text{ mono-}\chi) \\ &< 2^{k-1} \cdot 2^{1-k} = 1. \quad \square \end{aligned}$$

Note how this proof breaks down if $|\mathcal{F}| \geq 2^{k-1}$, and hence especially if $|\mathcal{F}|$ increases with the number of vertices n .

Now if the edges $S \in \mathcal{F}$ were disjoint (nonintersecting), then the events $A_S \sim "S \text{ mono-}\chi"$ would be mutually independent, and one would have

$$\Pr(\forall S \in \mathcal{F} : S \text{ not mono-}\chi) = \prod_{S \in \mathcal{F}} \Pr(S \text{ not mono-}\chi) = (1 - 2^{1-k})^{|\mathcal{F}|} > 0$$