Threshold functions

- The most interesting phenomena in $\mathcal{G}(n,p)$ random graphs emerge when p is not a constant but $p = p(n) \to 0$ in some controlled way.

- Recall: function $t = t(n)$ is a threshold for graph property A if

(i) $p \ll t \Rightarrow G \not\models A$ for a.e. $G \in \mathcal{G}(n,p)$,

(ii) $p \gg t \Rightarrow G \models A$ for a.e. $G \in \mathcal{G}(n,p)$.

- As an example, let us review Theorem 4.6: define the density of a graph $G = (V,E)$ as $\rho(G) = |E|/|V|$, and say that G is balanced if $\rho(G') \le \rho(G)$ for all subgraphs G' of G.

*) also Thm 4.6

- Theorem 5.8*) (Erdős & Rényi 1960). Let H be a balanced graph. Then the graph property "G has a subgraph isomorphic to H" has threshold $n^{-1/\rho(H)}$.

Proof. We apply the first- and second-moment methods as in the special case of Thm 4.3, but now simplify the calculations using Lemma Δ^* (p. 38).

For a given balanced graph H, denote $l = |E|$, $k = |V|$, so that $\rho(H) = l/k$.

(i) [Upper threshold / 1st-moment method:] For each vertex set S, $|S| = k$, define the indicator variable

$$X_S \sim \text{"}S \text{ contains a copy of } H\text{"}$$

and consider the sum $X = \sum_{|S|=k} X_S$.

Now

$$p^{\ell} \le \Pr(X_S = 1) \le k!\, p^{\ell}$$

(The upper bound is due to the fact that each ordering of the k vertices induces at most one copy of H.)

Thus, by linearity of expectation:

$$E[X] = \sum_{|S|=k} E[X_S] = \binom{n}{k} \Pr(X_S = 1)$$

$$= \Theta(n^k p^{\ell}).$$

Now if $p \ll n^{-k/\ell}$, then $E[X] \to 0$ as $n \to \infty$, and consequently also $\Pr(X>0) \to 0$ as $n \to \infty$.

(ii) [Lower threshold / 2nd-moment method:] Now assume that $p \gg n^{-k/\ell}$, so that $E[X] \to \infty$ as $n \to \infty$. By lemma Δ^*, in order to show that $\Pr(X>0) \to 1$ as $n \to \infty$, it suffices to show that $\Delta^* = o(E[X])$, where

$$\Delta^* = \sum_{T \sim S} \Pr\left(X_T = 1 \mid X_S = 1\right) \qquad \text{for fixed } S.$$

(Note that all the k-sets of vertices "look the same" except for the numbering of the vertices.)

Here $T \sim S$ iff $T \neq S$ and T & S have common edges, i.e. if $|T \cap S| = r$ for some $r = 2, \dots, k-1$. Thus:

$$\Delta^* = \sum_{r=2}^{k-1} \sum_{|T \cap S| = r} \Pr(X_T = 1 \mid X_S = 1).$$

Now for a fixed S and given r, there are $\binom{k}{r}\binom{n-k}{k-r}$ $= O(n^{k-r})$ choices of T.

For any choice of T, there are at most $k! = O(1)$ copies of H on T. Each of these contains at most

$$\varrho(H)\cdot r = \frac{r\ell}{k}$$

edges both of whose endpoints are also in S. (Consider the induced subgraph of H on $T \cap S$ and note that H is balanced!) – Consequently each copy of H on T contains at least $\ell - r\ell/k$ edges one of whose endpoints is not in S, and so

$$\Pr(X_T = 1 \mid X_S = 1) \leq k!\, p^{\ell - \frac{r\ell}{k}} = O(p^{\ell(1-r/k)})$$

Hence

$$\begin{aligned}
\Delta^* &= \sum_{r=2}^{k-1} \binom{k}{r}\binom{n-k}{k-r}\, O(p^{\ell(1-r/k)}) \\
&= \sum_{r=2}^{k-1} O(n^{k-r}\, p^{\ell(1-r/k)}) \\
&= \sum_{r=2}^{k-1} O((n^k p^\ell)^{1-r/k}) \qquad \Big|\; p \gg n^{-k/\ell} \Rightarrow n^k p^\ell \to \infty \\
&= O(k\cdot (n^k p^\ell)^{1-r/k}) \\
&= o(n^k p^\ell) \\
&= o(E[X]).
\end{aligned}$$

Lemma Δ^* thus applies, and $\Pr(X>0) \to 1$ as $n \to \infty$. $\square$