5. Random Graphs

- Recall: $\mathcal{G}_n$ = family of all (labelled, undirected) n-node graphs
  - $|\mathcal{G}_n| = 2^N$, where $N = \binom{n}{2}$
- The "Erdős - Rényi" ensembles of random graphs:
  - $\mathcal{G}(n,p)$ = all $G \in \mathcal{G}_n$ taken so that each edge has occurrence prob p, indep. of the other edges. [Gilbert 1959]
  - $\mathcal{G}(n,M)$ = all $G \in \mathcal{G}_n$ with exactly $M \leq N$ edges, taken with uniform probability. [Erdős & Rényi 1960]
- Since for large n and given p, the number of edges in a $\mathcal{G}(n,p)$ random graph is heavily concentrated around $M = Np$, the ensembles $\mathcal{G}(n,p)$ and $\mathcal{G}(n,M)$ for $M = Np$ are "very similar" (?). Ensemble $\mathcal{G}(n,p)$ is easier to analyse and thus usually considered.
- Denote $G \models A$ ~ "graph G has property A".
- Random graphs $G \in \mathcal{G}(n,p)$ have property A <u>asymptotically almost surely</u> (a.a.s.) or <u>almost everywhere</u> (a.e.) if

$$\Pr_{n,p}(G \models A) \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

- Denote in general $q = 1-p$.

- For constant p, $0<p<1$, the features of $\mathcal{G}(n,p)$ graphs are quite regular, as illustrated by the following results:

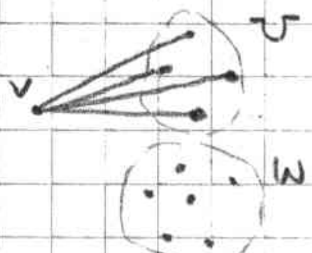
- <u>Theorem</u> 5.1 Let H be a fixed graph and $0<p<1$. Then a.e. $G \in \mathcal{G}(n,p)$ contains an induced copy of H.

  <u>Proof</u>. Let $k = |H|$ = number of vertices in H. Then a graph G with $n = |G| \geq k$ vertices can be partitioned into $\lfloor n/k \rfloor$ disjoint sets of k vertices (with some left over). For each of these sets, the prob. that it forms an induced copy of H is some $r>0$.

  Thus, the prob. that none of these sets forms an induced copy of H is

  $$(1-r)^{\lfloor n/k \rfloor} \to 0 \quad \text{as } n \to \infty. \;\square$$

- Let $k, \ell \in \mathbb{N}$. Say that a graph $G=(V,E)$ has property $Q_{k\ell}$, if $\forall U, W \in V$, $|U| \leq k$, $|W| \leq \ell$, $U \cap W = \emptyset$, G contains a vertex $v \in V \setminus (U \cup W)$ that is adjacent to all $u \in U$ and to no $w \in W$.



- <u>Lemma</u> 5.2 For every constant p, $0<p<1$, and all $k, \ell \in \mathbb{N}$, a.e. $G \in \mathcal{G}(n,p)$ has property $Q_{k\ell}$

  <u>Proof</u>. For fixed $U, W, v \in V \setminus (U \cup W)$, the prob. that the condition is satisfied is

  $$p^{|U|} q^{|W|} \geq p^k q^\ell .$$

  The events are independent for so the prob. that <u>no</u> appropriate v exists for <u>given</u> U, W is

  $$(1-p^{|U|} q^{|W|})^{n-|U|-|W|} \leq (1-p^k q^\ell)^{n-k-\ell}.$$

There are at most $n^{k+\ell}$ (U,W)-pairs to be considered, so the prob. that some pair has no good v is bounded by:

$$n^{k+\ell}\,(\underbrace{1-p^k q^\ell}_{<1})^{n-k-\ell} \longrightarrow 0 \quad \text{as} \quad n\to\infty.$$

Thus in a.e. $G\in\mathcal{G}(n,p)$ all (U,W) have some good v. $\square$

- Corollary 5.3 Let p, $0<p<1$, be constant. Then:

(i) for any constant k, a.e. $G\in\mathcal{G}(n,p)$ has min degree $\geq k$;

(ii) a.e. $G\in\mathcal{G}(n,p)$ has diameter 2;

(iii) for any constant k, a.e. $G\in\mathcal{G}(n,p)$ is k-connected.

Proof (i), (ii) Immediate from Lemma 5.2.

(iii) In a.e. $G\in\mathcal{G}(n,p)$, no two vertices u_1,u_2 can be separated by a cutset of size $k-1$, because one may choose in Lemma 5.2 $U=\{u_1,u_2\}$, $W=\{w_1,\dots,w_{k-1}\}$ for arbitrary $w_1,\dots,w_{k-1}$, and obtain a path u_1-v-u_2 connecting u_1,u_2 and avoiding $w_1,\dots,w_{k-1}$. $\square$

- More generally, it is possible to prove (Alon & Spencer Sec. 10.?):

Theorem 5.4 Let A be any first-order property of graphs (i.e. A is expressible in terms of quantification over vertices, edge relation $E(u,v)$, and identity). Then for any constant p, $0<p<1$, either $G\models A$ a.e. or $G\models \text{not } A$ a.e. $\square$

- This is called a zero-one law for random graphs (w.r.t. first-order properties and constant p).