3. The Alteration Method

- Idea: Instead of proving the existence of the desired "good" structure directly, prove the existence of an "almost-good" structure and account for the corrections needed to make it "good".

- Example 1. Ramsey numbers.

Recall:

$R(k)$ = smallest n_k s.th
$n \geq n_k \to$ any two-colouring of edges of K_n contains a mono-χ K_k.

- Theorem 3.1 For any integer n,

$$R(k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

Proof. For any given n, Consider a random (unif., indep.) two-colouring of K_n. For any subset of k vertices R, define indicator variable

$X_R \sim$ "R is monochromatic".

Then $X = \sum_R X_R$ counts the number of mono-χ K_k's and

$$E[X] = \sum_R E[X_R] = \binom{n}{k} \cdot 2^{1-\binom{k}{2}} \triangleq m.$$

Thus, for some two-colouring ξ of K_n, $X(\xi) \leq m$.

Now remove from K_n one vertex from each K_k that is mono-χ under ξ. Of course the same vertex may be marked for removal many times because of different K_k's, but in any case at most m vertices are removed. Now when restricted to the remaining K_s, $s \geq n-m$, colouring ξ contains no more mono-χ K_k's. $\square$

- Note. In the bound $g_k(n) = n - \binom{n}{k} 2^{1-\binom{k}{2}}$, $g_k(0)=0$, $g_k(n) \to -\infty$ as $n \to \infty$. An optimal choice of n is
$$n \sim \frac{1}{e} k 2^{k/2} (1-o(1)),$$
yielding a bound $R(k) > \frac{1}{e}(1+o(1)) k 2^{k/2}$.

[Quick estimate: for $k \ll n$, $\binom{n}{k} \sim \left(\frac{ne}{k}\right)^k$, $2^{1-\binom{k}{2}} \sim 2^{k^2/2}$.
Let's solve approximately for $g_k(n) > 0$:
$$\left(\frac{ne}{k}\right)^k \cdot 2^{-k^2/2} = 1 \Leftrightarrow \frac{ne}{k} \cdot 2^{-k/2} = 1 \Leftrightarrow n = \frac{k}{e} \cdot 2^{k/2}.$$]

- Also off-diagonal Ramsey numbers are often considered:

$R(k,\ell)$ = smallest $n_{k\ell}$ s.th. red/blue
$n \geq n_{k\ell} \to$ any two-colouring of edges of K_n contains either a red K_k or a blue K_ℓ.

The basic probabilistic method (cf. Thm 1.1) yields: or linearity of expectation

Theorem 3.2 If for some $p \in [0,1]$:
$$\binom{n}{k} p^{\binom{k}{2}} + \binom{n}{\ell}(1-p)^{\binom{\ell}{2}} < 1,$$
then $R(k,\ell) > n$. □

By the alteration method, one can prove:

Theorem 3.3 For any integer n and $p \in [0,1]$:
$$R(k,\ell) > n - \binom{n}{k} p^{\binom{k}{2}} - \binom{n}{\ell}(1-p)^{\binom{\ell}{2}}$$

Proof Consider random red-blue colouring of K_n. Compute exp. numbers of red K_k's and blue K_ℓ's. Remove one vertex from each, yielding a non-mono-χ colouring on a smaller K_s. □

- Example 2. Independent sets.

For a graph $G = (V, E)$, the independence number is

$$\alpha(G) = \max\{|S| : S \subseteq V, \text{ no two vertices in } S \text{ are connected by an edge in } E\}.$$

- Theorem 3.4 Let $G = (V, E)$, $|V| = n$, $d = \frac{2|E|}{n} \geq 1$ (avg degree). Then

$$\alpha(G) \geq \frac{n}{2d}.$$

Note This in one half of so called Turán's theorem (1941). The other half gives a structural characterisation of graphs with given $\alpha(G)$ and minimal number of edges (→ "extremal graph theory").

Proof. For a given (to be determined) $p \in [0,1]$, consider a random (indep.) set of vertices $S \subseteq V$ defined by:

$$\Pr(v \in S) = p.$$

Assess the exp. number of vertices X and edges Y in S:

$$E[X] = np, \quad E[Y] = |E| \cdot p^2 = \frac{nd}{2} \cdot p^2.$$

Thus, by setting $p = \frac{1}{d}$:

$$E[X - Y] = np - \frac{nd}{2} p^2 = \frac{n}{d} - \frac{n}{2d} = \frac{n}{2d},$$

i.e. some $S \subseteq V$ contains $\geq \frac{n}{2d}$ many more vertices than edges. Removing one endpoint of each edge in S results in a set $S' \subseteq V$, $|S'| \geq \frac{n}{2d}$, where no two vertices are connected by an edge. $\square$