T-79.5201 Discrete Structures, Autumn 2007

Tutorial 10, 12 December

1. Let S be a finite set of points chosen from the lattice $\mathbf{Z} \times \mathbf{Z}$. Prove that there exists a red/blue-colouring of the points in S so that in every horizontal and vertical line the number of red and blue points differs by at most three. (In fact, one can colour the points in S so that the difference is at most *one*. Can you prove this more difficult result?)

2. Give a recursive construction for producing Hadamard matrices of order 2^k for all $k \geq 1$. (*Hint:* First construct a Hadamard matrix of order 2, say H_2. Then construct a Hadamard matrix of order 4 as a 2×2 block matrix, with H_2 blocks. Generalise.)

3. [Alon & Spencer, Prob. 12.1:] Let $\mathcal{A}$ be a family of m subsets of $[n]$, with n even. Let $\chi(i)$, $i = 1, \ldots, \frac{n}{2}$ be independent and uniform in $\{-1, +1\}$ and set $\chi(i + \frac{n}{2}) = -\chi(i)$ for $i = 1, \ldots, \frac{n}{2}$. Using this random colouring, improve Theorem 9.1 by showing

$$\mathrm{disc}(\mathcal{A}) \leq \sqrt{\frac{n}{2} \ln(2m)}.$$