T-79.5201 Discrete Structures, Autumn 2007

Tutorial 9, 5 December

1. (a) Prove that if $X_0, \dots, X_n$ is a martingale, then $E[X_n] = E[X_0]$. (*Hint:* $E[Y] = E[E[Y|Z]]$.)

   (b) Prove, as a consequence to Azuma's inequality (Theorem 8.2 in the lecture notes), that if $X_0, \dots, X_n$ is a martingale with $|X_{k+1} - X_k| \leq 1$ for all $k = 0, \dots, n-1$, then for any $a > 0$, $\Pr(|X_n - X_0| > a) < 2e^{-a^2/2n}$.

2. (a) Define the edge exposure martingale for the independence number $\alpha(G)$ of $\mathcal{G}(n,p)$ random graphs, and draw a similar tree diagram as on p. 68 of the lecture notes (p. 94 in Alon & Spencer) to illustrate the computation of the values $X_k(H)$ for this martingale in the case $n = 3$, $p = \frac{1}{2}$.

   (b) Verify that the sequence of random variables $X_0, \dots, X_n$ defined in part (a) of the problem indeed satisfies the martingale condition.

3. Consider the task of placing n balls in n bins independently at random. Denote by F_n the number of remaining free (= empty) bins after all the balls have been placed.

   (a) Calculate the value $f_n = E[F_n]$.

   (b) Establish some bound on the probability $\Pr(|F_n - f_n| > a)$, for $a > 0$. (*Hint:* Denote by $g, h : [n] \to [n]$ arbitrary assignments of balls into bins, and consider the "bin exposure martingale" defined as

$$X_k(h) = E[F_n(g) \mid g(i) = h(i) \text{ for } i = 1, \dots, k\ ].)$$