1. Let \( n \geq 1 \). A family \( \mathcal{U} \) of subsets of \([n] = \{1, \ldots, n\}\) is an **upset** or **filter** if \( A \in \mathcal{U}, A \subseteq B \implies B \in \mathcal{U} \). Similarly, \( \mathcal{D} \subseteq \mathcal{P}([n]) \) is a **downset** or **ideal** if \( A \in \mathcal{D}, B \subseteq A \implies B \in \mathcal{D} \).

   Prove **Kleitman’s Lemma**: If \( \mathcal{U} \) is an upset on \([n]\) and \( \mathcal{D} \) is a downset on \([n]\), then:
   \[ |\mathcal{U}| \cdot |\mathcal{D}| \geq 2^n |\mathcal{U} \cap \mathcal{D}|. \]

2. A family \( \mathcal{A} \) of subsets of \([n]\) is **intersecting**, if \( A \cap B \neq \emptyset \) for any \( A, B \in \mathcal{A} \). Prove the following claims:

   (a) For any intersecting family \( \mathcal{A} \) on \([n]\), \( |\mathcal{A}| \leq 2^{n-1} \).

   (b) For any intersecting family \( \mathcal{A} \) on \([n]\) that moreover satisfies \( A \cup B \neq [n] \) for any \( A, B \in \mathcal{A} \), \( |\mathcal{A}| \leq 2^{n-2} \). (Hint: Observe that for any family of sets \( \mathcal{A} \), there exist an upset \( \mathcal{U} \) and a downset \( \mathcal{D} \) such that \( \mathcal{A} = \mathcal{U} \cap \mathcal{D} \).)

   (c) The bounds on the size of \( \mathcal{A} \) in both (a) and (b) are best possible, i.e. there are families \( \mathcal{A} \) that achieve these bounds.

3. [Alon & Spencer, Prob. 6.3:] Show that the probability that in a random graph \( G \in \mathcal{G}(2k, 1/2) \), the maximum degree of any vertex is at most \( k - 1 \), is at least \( 1/4^k \).