T-79.5201 Discrete Structures, Autumn 2007
Home assignment 2 (due 5 Dec at 12:15 p.m.)

1. Prove that the graph property "G contains a d-dimensional cube" has a threshold function for $d \geq 1$, and compute it. (The "d-dimensional cube" has 2^d vertices represented as $\{0,1\}^d$, and two vertices are connected by an edge if and only if their representations differ in exactly one position.)

2. Show that for $p(n) = (1-\epsilon)(\ln n)/n$, $0 < \epsilon \leq 1$, almost every $G \in \mathcal{G}(n,p)$ contains an isolated vertex.

3. Let F be a Boolean formula in k-conjunctive normal form, i.e. such that each clause of F contains exactly k literals. Assume that each variable appears (negated or not) in at most r clauses of F, where $r \leq 2^{k-2}/k$. Prove that then there is a truth assignment to the variables that satisfies all the clauses. (*Hint:* Lovász Local Lemma.)

4. Let $n \geq 1$, and let μ and ν be probability distributions on $\Omega = \mathcal{P}([n])$ such that for all $A, B \subseteq [n]$, $\mu(A)\nu(B) \leq \mu(A \cup B)\nu(A \cap B)$. Show that for any increasing function $h : \Omega \to \mathbf{R}^+$,

$$\sum_{A \in \Omega} \mu(A)h(A) \geq \sum_{A \in \Omega} \nu(A)h(A).$$

   Observe, as a corollary, that under these conditions for any monotone increasing event $\mathcal{A} \subseteq \Omega$, $\Pr_\mu(\mathcal{A}) \geq \Pr_\nu(\mathcal{A})$.