1. Verify the following properties of the Riemann–Stieltjes integral, assuming that the integrals appearing in the formulas are well defined:

(a) Linearity:
\[
\int_a^b (c_1 f_1 + c_2 f_2) \, dg = c_1 \int_a^b f_1 \, dg + c_2 \int_a^b f_2 \, dg,
\]
\[
\int_a^b f \, d(c_1 g_1 + c_2 g_2) = c_1 \int_a^b f \, dg_1 + c_2 \int_a^b f \, dg_2.
\]

(b) Reduction to Riemann integral: for a continuously differentiable function \( g \),
\[
\int_a^b f(t) \, dg(t) = \int_a^b f(t)g'(t) \, dt.
\]

2. Let \( f \) and \( g \) be continuous functions and \( a, b \in \mathbb{Z} \). Verify the correctness of the following formulas, assuming that the integrals contained in them are well defined:

\[
\int_a^b f(t) \, dg([t]) = \sum_{a \leq k < b} f(k) \Delta g(k), \quad \Delta g(k) = g(k + 1) - g(k);
\]
\[
\int_a^b f([t]) \, dg(t) = \sum_{a < k \leq b} f(k) \nabla g(k), \quad \nabla g(k) = g(k) - g(k - 1).
\]

Derive from the preceding formulas the following “partial summation rule”:
\[
\sum_{a \leq k < b} f(k) \Delta g(k) = \int_a^b f(k)g(k) \, dt - \sum_{a < k \leq b} g(k) \nabla f(k).
\]

3. Use Euler’s summation formula to estimate the following sums:

(a) Sum \( \sum_{1 \leq k < n} k^{1/2} \) up to order \( O(1) \).
(b) Sum \( \sum_{1 \leq k < n} k^r \), \( r \in \mathbb{N} \), exactly.