1. Determine algebraic expressions for the following generating functions, based directly on the structure of the respective combinatorial families:

(a) The egf for sequence \( (a_n) \), where \( a_n \) is the number of ways an \( n \)-element ground set can be partitioned into (unordered) pairs of exactly two elements. (In other words, \( a_n \) is the number of perfect matchings of a complete \( n \)-node graph.)

(b) The egf for sequence \( (b_n) \), where \( b_n \) is the number of “binary tree partitions” of the set \( [n] = \{1, \ldots, n\} \), i.e. the number of labelled binary trees where each node contains some nonempty subset of the set \( [n] \), these subsets are disjoint and together cover all of \( [n] \). (A binary tree is an ordered rooted tree, where each nodes has two descendant subtrees, either or both of which may be empty. By direct drawing and counting one observes that \( b_0 = 1 \), \( b_1 = 1 \), \( b_2 = 5 \), \( b_3 = 43 \) etc.)

2. Show that if a combinatorial family \( B \) can be decomposed as \( B \rightarrow A^{[a]} \), then the counts of objects in families \( B \) and \( A \) of different weights are related by:

\[
nb_n = \sum_{k=0}^{n} \binom{n}{k} k a_k b_{n-k}.
\]

(Hint: The “zD log” trick.) Based on this result, derive a recurrence formula for the number \( c_n \) of connected labelled graphs with \( n \) nodes, and use your formula to compute the values \( c_1, \ldots, c_6 \). (Hint: Determine first the total number \( g_n \) of all \( n \)-node graphs.)

3. Determine the number of strings of length \( n \) generated by the context free grammar

\[ S \rightarrow aSS | bS | cS | d \]

(If you are not familiar with the grammar formalism, please consult the course personnel.)

4. Estimate the value of the sum \( \sum_{k=1}^{n} k \ln k \) up to order \( O(1) \). (Hint: Consider first the sum with an upper bound of \( n - 1 \) instead of \( n \).) What estimate can you derive from this for the rate of growth of the product \( 1^1 \cdot 2^2 \cdots n^n \) as a function of \( n \)?