# **NP**-COMPLETE PROBLEMS

- Characterizing **NP**
- Variants of satisfiability
- Graph-theoretic problems
- Coloring problems
- Sets and numbers
- Pseudopolynomial algorithms

(C. Papadimitriou: *Computational complexity*, Chapter 9)





## 1. Characterizing NP

**Definition.**

1. A relation $R \subseteq \Sigma^* \times \Sigma^*$ is *polynomially decidable* iff there is a deterministic TM deciding the language $\{x;y \mid (x,y) \in R\}$ in polynomial time.
2. A relation $R$ is *polynomially balanced* if $(x,y) \in R$ implies $|y| \leq |x|^k$ for some $k \geq 1$.

**Proposition.** Let $L \subseteq \Sigma^*$ be a language.

Now $L \in \mathbf{NP}$ iff there is a polynomially balanced and polynomially decidable relation $R$ such that $L = \{x \in \Sigma^* \mid (x,y) \in R \text{ for some } y \in \Sigma^*\}$.





## Proof

($\Leftarrow$) Suppose there is such a relation $R$. Then $L$ is decided by a NTM that on input $x$, guesses a $y$ of length at most $|x|^k$ and uses the machine for $R$ to decide in polynomial time whether $(x,y) \in R$.

($\Rightarrow$) Suppose that $L \in \mathbf{NP}$, i.e. there is a NTM $N$ deciding $L$ in time $|x|^k$ for some $k$.

Define a relation $R$ as follows: $(x,y) \in R$ iff $y$ is the encoding of an accepting computation of $N$ on input $x$. Now $R$ is polynomially

- balanced (each computation of $N$ is polynomially bounded) and
- decidable (since it can be checked in linear time whether $y$ encodes an accepting computation of $N$ on $x$).
- As $N$ decides $L$, $L = \{x \mid (x,y) \in R \text{ for some } y\}$. $\square$





## Succinct certificates

A problem is in **NP** if any "yes" instance $x$ of the problem has at least one **succinct certificate**, or polynomial witness, $y$. **NP** contains a huge number of practically important, natural computational problems:

- A typical problem is to construct a mathematical object satisfying certain specifications (path, solution of equations, routing, VLSI layout,...). This is the certificate.
- The decision version of the problem is determine whether at least one such an object exists for the input.
- The object is usually not very large compared to the input.
- The specifications of the object are usually simple enough to be checkable in polynomial time.

## Boundary between NP and P

- Most problems arising in computational practice are in **NP**.
- Computational complexity theory provides us tools to study which problems in **NP** belong to **P** and which do not.
- **NP**-completeness is a basic tool in this respect:
  Showing that a problem is **NP**-complete implies that the problem is among the least likely to be in **P**.
  (If an **NP**-complete problem is in **P**, then **NP** = **P**.)





## NP-completeness and algorithm design techniques

When a problem is known to be **NP**-complete, further efforts are usually directed to:

(i) Attacking special cases

(ii) Approximation algorithms

(iii) Studying average case performance

(iv) Randomized algorithms

(v) (Exponential) algorithms that are practical for small instances

(vi) Local search methods





# 2. Variants of Satisfiability

- Many problems if generalized enough become **NP**-complete.
- Often it is important to find the dividing line between **P** and **NP**-completeness.
- One basic technique is to investigate the set of instances produced by a reduction $R$ involved in the **NP**-completeness proof in order to capture another **NP**-complete problem.
- Next we consider variants of SAT such as

  3SAT, 2SAT, MAX2SAT, and NAESAT

  and analyze their computational complexities.





## $k$SAT problems

**Definition.** $k$SAT, where $k \geq 1$ is an integer, is the set of Boolean expressions $\phi \in$ SAT (in CNF) whose all clauses have exactly $k$ literals.

**Proposition.** 3SAT is **NP**-complete.

Proof.

- 3SAT is in **NP** as a special case of SAT which is in **NP**.
- CIRCUIT SAT was shown to be **NP**-complete and a reduction from CIRCUIT SAT to SAT has already been presented.
- Consider now the clauses in the reduction. They have all at most 3 literals. Each clause with one or two literals can be modified to an equivalent clause with exactly 3 literals by duplicating literals.
- Hence, we can reduce CIRCUIT SAT to 3SAT. □

## Narrowing NP-complete languages

- An **NP**-complete languages can sometimes be narrowed down by *transformations* which eliminate certain features of the language but still preserve **NP**-completeness.
- The following result is a typical example.

**Proposition.** 3SAT remains **NP**-complete even if each variable is restricted to appear at most three times in a Boolean expression $\phi \in$ 3SAT and each literal at most twice in $\phi$.

Proof. This is shown by a reduction where any instance $\phi$ of 3SAT is rewritten to eliminate the forbidden features.





## Proof

- Consider a variable $x$ appearing $k > 3$ times in $\phi$.
  (i) Replace the first occurrence of $x$ in $\phi$ by $x_1$, the second by $x_2$, and so on where $x_1, \ldots, x_k$ are new variables.
  (ii) Add clauses $(\neg x_1 \vee x_2), (\neg x_2 \vee x_3), \ldots, (\neg x_k \vee x_1)$ to $\phi$.
- Let $\phi'$ be the expression $\phi$ modified systematically in this way.
- It follows that $\phi'$ has the desired syntactic properties.
- Now $\phi$ is satisfiable iff $\phi'$ is satisfiable:
  For each $x$ appearing $k > 3$ times in $\phi$, the truth values of $x_1, \ldots, x_k$ are the same in each truth assignment satisfying $\phi'$. □





## Boundary between P and NP-completeness

- The boundary is between 2SAT and 3SAT.
- For an instance $\phi$ of 2SAT, there is a polynomial time algorithm which is based on reachability in a graph associated with $\phi$.

**Definition.** Let $\phi$ be an instance of 2SAT.

Define a graph $G(\phi)$ as follows:

– The variables of $\phi$ and their negations form the vertices of $G(\phi)$.

– There is an arc $(\alpha, \beta)$ iff there is a clause $\overline{\alpha} \vee \beta$ or $\beta \vee \overline{\alpha}$ in $\phi$, i.e., if $(\alpha, \beta)$ is an arc, so is $(\overline{\beta}, \overline{\alpha})$ where $\overline{\alpha}$ is the complement of $\alpha$.

**Theorem.** Let $\phi$ be an instance of 2SAT.

Then $\phi$ is unsatisfiable iff there is a variable $x$ such that there are paths from $x$ to $\neg x$ and from $\neg x$ to $x$ in $G(\phi)$.





## The complexity of 2SAT—cont'd

**Corollary.** 2SAT is in **NL** $(\subseteq \mathbf{P})$.

Proof. Since **NL** is closed under complement, it is sufficient to show that 2SAT COMPLEMENT is in **NL**.

The reachability condition of the preceding theorem can be tested in logarithmic space non-deterministically by guessing a variable $x$ and paths from $x$ to $\neg x$ and back. □

MAX2SAT is a generalization of 2SAT:

INSTANCE: a Boolean expression $\phi$ in CNF (having at most two literals per clause) and an integer bound $K$.
QUESTION: Is there a truth assignment satisfying at least $K$ clauses?

**Theorem.** MAX2SAT is **NP**-complete. (See tutorials.)

## The case of not-all-equal SAT (NAESAT)

For each $\phi \in$ NAESAT $\subset$ 3SAT, there is a truth assignment so that the three literals in each clause of $\phi$ do not have the same truth value.

**Theorem.** NAESAT is **NP**-complete.

Proof.

- ➤ CIRCUIT SAT was shown to be **NP**-complete and a reduction $R$ from CIRCUIT SAT to SAT has already been presented such that for a circuit $C$, $C \in$ CIRCUIT SAT iff $R(C) \in$ SAT.
- ➤ For all one- and two-literal clauses in the resulting set of clauses $R(C)$, add the same literal, say $z$, to make them 3-literal clauses.

Claim: it holds for the resulting Boolean expression $R_z(C)$ in 3CNF:

$$R_z(C) \in \text{NAESAT} \textit{ iff } C \in \text{CIRCUIT SAT}.$$





($\Rightarrow$) If a truth assignment $T$ satisfies $R_z(C)$ in the sense of NAESAT, so does the complementary truth assignment $\overline{T}$.

Thus, $z$ is **false** in either $T$ or $\overline{T}$ which implies that $R(C)$ is satisfied by $T$ or $\overline{T}$. Thus $C$ is satisfiable.

($\Leftarrow$) If $C$ is satisfiable, then there is a truth assignment $T$ satisfying $R(C)$. Let us then extend $T$ for $R_z(C)$ by assigning $T(z) =$ **false**.

In no clause of $R_z(C)$ all literals are **true** (they cannot be all **false**):

(i) Clauses for **true**/**false**/NOT/variable gates contain $z$ that is **false**.

(ii) For AND gates the clauses are: $(\neg g \vee h \vee z)$, $(\neg g \vee h' \vee z)$, $(g \vee \neg h \vee \neg h')$ where in the first two $z$ is **false**, and in the third all three cannot be true as then the first two would be not true.

(iii) The case of OR gates is similar. □





## 3. Graph-Theoretic Problems

- ➤ In this section, we will consider only undirected graphs $G = (V, E)$ and their properties.
- ➤ For instance, consider the problem of finding an *independent* subset $I$ of $V$, i.e., a set $I$ such that for all $i, j \in I$, $[i, j] \notin E$.

INDEPENDENT SET:
INSTANCE: An undirected graph $G = (V, E)$ and an integer $K$.
QUESTION: Is there an independent set $I \subseteq V$ with $|I| = K$.

**Theorem.** INDEPENDENT SET is **NP**-complete. (See tutorials.)

The subclass of graphs needed in the reduction implies the following:

**Corollary.** 4-DEGREE INDEPENDENT SET is **NP**-complete.





## Graph problems: CLIQUE and NODE COVER

- ➤ The problems in graph theory are often closely related; suggesting even trivial reductions between problems.

**Example.** Consider the following two graph theoretic problems:

CLIQUE:
INSTANCE: An undirected graph $G = (V, E)$ and an integer $K$.
QUESTION: Is there a clique $C \subseteq V$ with $|C| = K$?
(A set $C \subseteq V$ is clique iff for any two vertices $i, j \in I$, $[i, j] \in E$)

NODE COVER:
INSTANCE: An undirected graph $G = (V, E)$ and an integer $B$.
QUESTION: Is there a set $C \subseteq V$ with $|C| \leq B$ such that for all $[i, j] \in E$, $i \in C$ or $j \in C$?

## Trivial reductions for CLIQUE and NODE COVER

- Independent sets are closely related to cliques and node covers:

  A set $I \subseteq V$ of vertices is

  1. an independent set of $G$ iff it is a clique of the *complement* graph $\overline{G}$, and
  2. an independent set of $G$ iff $V - I$ is a node cover of $G$.

- Thus an instance $G; K$ of INDEPENDENT SET can be reduced to
  - an instance $\overline{G}; K$ of CLIQUE, and
  - an instance $G; |V| - K$ of NODE COVER.

**Corollary.** CLIQUE and NODE COVER are **NP**-complete.





## Graph problems: MIN CUT and MAX CUT

- A cut in an undirected graph $G = (V, E)$ is a partition of the nodes into two nonempty sets $S$ and $V - S$.
- The size of a cut is the number of edges between $S$ and $V - S$.
- The problem of finding a cut with the smallest size is in **P**:

  (i) The smallest cut that separates two given nodes $s$ and $t$ equals to the maximum flow from $s$ to $t$.

  (ii) Minimum cut: find the maximum flow between a fixed $s$ and all other nodes and choose the smallest value found.

- However, the problem of deciding whether there is a cut of a size greater than or equal to $K$ (MAX CUT) is much harder:

**Theorem.** MAX CUT is **NP**-complete.





## Reduction from NAESAT to MAX CUT

The **NP**-completeness of MAX CUT is shown for graphs with multiple edges between nodes by a reduction from NAESAT.

- For a conjunction of clauses $\phi = C_1 \wedge \ldots \wedge C_m$, we construct a graph $G = (V, E)$ so that

  $G$ has a cut of size 5m iff $\phi$ is satisfied in the sense of NAESAT.

- The nodes of $G$ are $x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n$ where $x_1, \ldots, x_n$ are the variables in $\phi$.
- The edges in $G$ include a triangle $[\alpha, \beta, \gamma]$ for each clause $\alpha \vee \beta \vee \gamma$ and $n_i$ copies of the edge $[x_i, \neg x_i]$ where $n_i$ is the number of occurrences of $x_i$ or $\neg x_i$ in the clauses.





## Correctness of the reduction

- Suppose there is a cut $(S, V - S)$ of size $5m$ or more.
- All variables can be assumed separate from their negations:

  If both $x_i, \neg x_i$ are on the same side, they contribute at most $2n_i$ edges to the cut and, hence, changing the one with fewer neighbors does not decrease the size of the cut.

- Let $S$ be the set of true literals and $V - S$ those false.
- The total number of edges in the cut joining opposite literals is $3m$. The remaining $2m$ are coming from triangles meaning that all $m$ triangles are cut, i.e. $\phi$ is satisfied in the sense of NAESAT.
- Conversely, a satisfying truth assignment (in the sense of NAESAT) gives rise to a cut of size $5m$. □

## Graph problems: MAX BISECTION

- In applications of graph partitioning, the sizes of $S$ and $V-S$ cannot be arbitrarily small or large.

  MAX BISECTION is the problem of determining whether there is a cut $(S,V-S)$ with size of $K$ or more such that $|S|=|V-S|$.
- Is MAX BISECTION easier than MAX CUT?

**Lemma.** MAX BISECTION is **NP**-complete.

Proof. Reducing MAX CUT to MAX BISECTION by modifying input:

Add $|V|$ disconnected new nodes to $G$. Now every cut of $G$ can be made a bisection by appropriately splitting the new nodes.

Now $G=(V,E)$ has a cut $(S,V-S)$ with size of $K$ or more iff the modified graph has a cut with size of $K$ or more with $|S|=|V-S|$. □





## Graph problems: BISECTION WIDTH

- The respective minimization problem, i.e. MIN CUT with the bisection requirement, is **NP**-complete, too.
  (Remember that MIN CUT $\in$ **P**).
- BISECTION WIDTH: is there a bisection of size $K$ or less?

**Theorem.** BISECTION WIDTH is **NP**-complete.

Proof. A reduction from MAX BISECTION:

A graph $G=(V,E)$ where $|V|=2n$ for some $n$ has a bisection of size $K$ or more iff the complement $\overline{G}$ has a bisection of size $n^2-K$ or less. □





## General instructions how to establish NP-completeness

Designing an **NP**-completeness proof for a given problem $Q$

- Work on small instances of $Q$ to develop gadgets/primitives.
- Look at known **NP**-complete problems.
- Design a reduction $R$ **from** *a known **NP**-complete problem* **to** $Q$.
- Typical ingredients of a reduction:

  choices + consistency + constraints.
- The key question is how to express these in $Q$?





## Graph problems: HAMILTON PATH

**Theorem.** HAMILTON PATH is **NP**-complete.

Proof.

- Reduction from 3SAT to HAMILTON PATH:

  given a formula $\phi$ in CNF with variables $x_1,\ldots,x_n$ and clauses $C_1,\ldots,C_m$ each with three literals, we construct a graph $R(\phi)$ that has a Hamilton path iff $\phi$ is satisfiable.
- *Choice gadgets* select a truth assignment for variables $x_i$.
- *Consistency gadgets* (XOR) enforce that all occurrences of $x_i$ have the same truth value and all occurrences of $\neg x_i$ the opposite.
- *Constraint gadgets* guarantee that all clauses are satisfied.

## Gadgets [Papadimitriou, 1994]

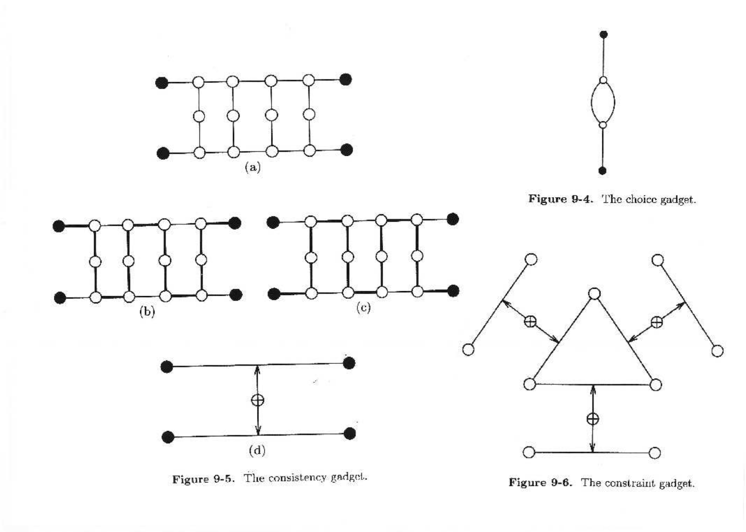



Figure 9-4. The choice gadget.

Figure 9-5. The consistency gadget.

Figure 9-6. The constraint gadget.





## Reduction from 3SAT to HAMILTON PATH

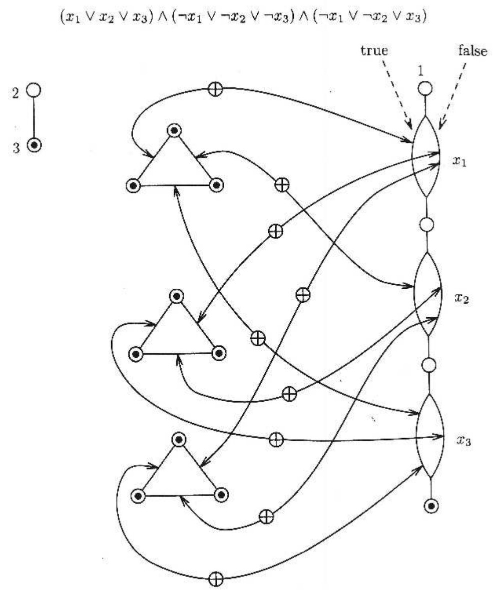
The graph $R(\phi)$ is constructed as follows:

- The *choice gadgets* of variables $x_i$ are connected in series.
- A *constraint gadget* (triangle) for each clause with an edge identified with each literal $l$ in the clause.
  - If $l$ is $x_i$, then XOR to **true** edge of choice gadget of $x_i$.
  - If it is $\neg x_i$, then XOR to **false** edge of choice gadget of $x_i$.
- All nodes of the triangles, the end node of choice gadgets and a new node 3 form a clique. Add a node 2 connected to 3.

**Basic idea**: each side of the constraint gadget is traversed by the Hamilton path iff the corresponding literal is **false**. Hence, at least one literal in any clause is **true** since otherwise all sides for its triangle should be traversed which is impossible (implying no Hamilton path).



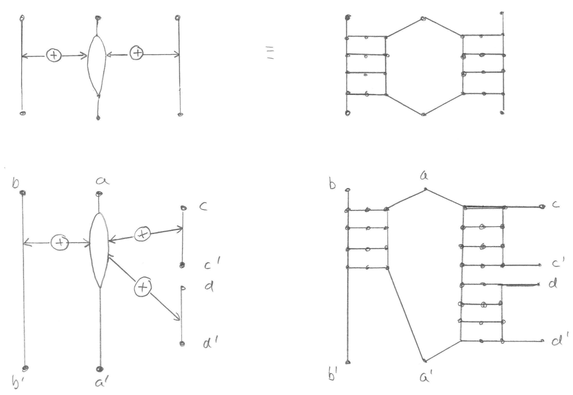





[Papadimitriou, 1994]







## Correctness of the reduction

- If there is a Hamilton path, $\phi$ is satisfiable:

  The path starts at 1, makes a truth assignment, traverses the triangles in some order and ends up in 2. The truth assignment satisfies $\phi$ as there are no triangle where all sides are traversed, i.e., where all literals are **false**.

- If $\phi$ is satisfiable, there is a Hamilton path:

  From a satisfying truth assignment, we construct a Hamilton path by starting at 1, traversing choice gadgets according to the truth assignment, the rest is a big clique for which a trivial path can be found leading to 3 and then to 2. $\Box$





## Travelling salesperson (TSP) revisited

**Corollary.** TSP(D) is **NP**-complete.

Proof: A reduction from HAMILTON PATH to TSP(D). Given a graph $G$ with $n$ nodes, construct a distance matrix $d_{ij}$ and a budget $B$ so that there is a tour of length $B$ or less iff $G$ has a Hamilton path.

- There are $n$ cities and the distance $d_{ij} = 1$ if there is $[i, j] \in G$ and $d_{ij} = 2$ otherwise. The budget $B = n+1$.
- If there is a tour of length $n+1$ or less, then there is at most one pair $(\pi(i), \pi(i+1))$ in it with cost 2, i.e., a pair for which $[\pi(i), \pi(i+1)]$ is not an edge. Removing it gives a Hamilton path.
- If $G$ has a Hamilton path, then its cost is $n-1$ and it can be made a tour with additional cost of 2. $\Box$





## 4. Coloring Problems

Consider the following problem:

k-COLORING:
INSTANCE: An undirected graph $G = (V, E)$.
QUESTION: Is there an assignment of one of $k$ colors to each of the nodes in $V$ such that any two nodes $i, j$ connected by an edge $[i, j] \in E$ do not have the same color?

☞ Coloring with $k = 2$ colors is easy (in **P**) but not when $k = 3$.





## Determining the complexity of 3-COLORING

**Theorem.** 3-COLORING is **NP**-complete.

Proof. A reduction from NAESAT to 3-COLORING.

- For a conjunction clauses $\phi = C_1 \wedge \ldots \wedge C_m$ with variables $x_1, \ldots, x_n$, construct a graph $G(\phi)$ that can be colored with $\{0, 1, 2\}$ iff there is a truth assignment satisfying all clauses in the way of NAESAT.
- *Choice gadgets:* For each variable $x_i$, we introduce a triangle $[a, x_i, \neg x_i]$, i.e. all triangles share a node $a$.
- *Constraints:* For each clause $C_i$: a triangle $[C_{i1}, C_{i2}, C_{i3}]$ where each $C_{ij}$ is further connected to the node with the $j$th literal in $C_i$.

[Papadimitriou, 1994]

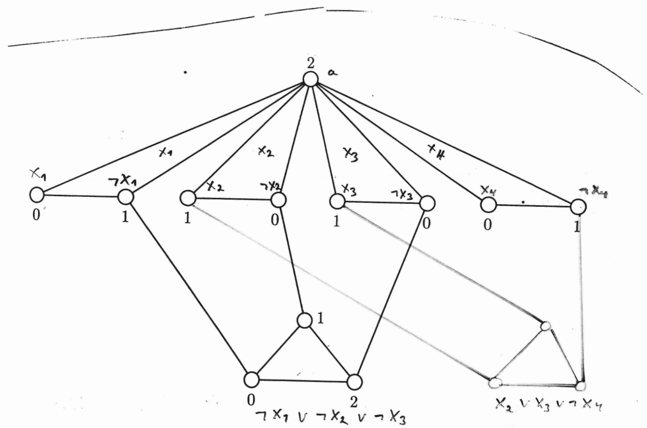


**Figure 9-8.** The reduction to 3-COLORING.





## Correctness of the reduction

($\Rightarrow$) Suppose $G$ can be colored with $\{0,1,2\}$ and $a$ has color 2. This induces a truth assignment $T$ via the colors of the nodes $x_i$: if the color is 1, then $T(x_i) = \textbf{true}$ else $T(x_i) = \textbf{false}$.

If we assume that $T$ assigns all literals of some clause $C_i$ to true/false, then color 1/0 cannot be used for coloring $[C_{i1}, C_{i2}, C_{i3}]$, a contradiction. Thus $\phi$ is satisfied in the sense of NAESAT.

($\Leftarrow$) Assume that $\phi$ is satisfied by $T$ in the sense of NAESAT. Then we can extract a coloring for $G$ from $T$ as follows:

1. Node $a$ is colored with color 2.
2. If $T(x_i) = \textbf{true}$, then color $x_i$ with 1 and $\neg x_i$ with 0 else vice versa.
3. From each $[C_{i1}, C_{i2}, C_{i3}]$, color two literals having opposite truth values with 0 (**true**) and 1 (**false**). Color the third with 2. □





# 5. Sets and Numbers

TRIPARTITE MATCHING:
INSTANCE: Three sets $B$ (boys), $G$ (girls), and $H$ (homes) each containing $n$ elements and a ternary relation $T \subseteq B \times G \times H$.
QUESTION: Is there a set of $n$ triples in $T$ no two of which have a component in common?

**Theorem.** TRIPARTITE MATCHING is **NP**-complete.

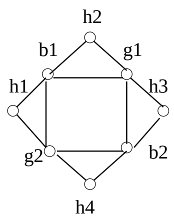
Proof. By a reduction from 3SAT. Each variable $x$ has a combined choice and consistency gadget and each clause $c$ a dedicated pair of a boy $b_c$ and a girl $g_c$ and three triples $(b_c, g_c, h_l)$ where $h_l$ ranges over the three homes corresponding to the occurrences of literals in the clause (appearing in the combined gadgets).





## The combined gadget for choice and consistency

The gadget for a variable $x$ involves $k$ boys and $k$ girls forming a $k$ circle and $2k$ homes where $k$ is either the number of occurrences of $x$ or its negation whichever is larger. (Recall that $k$ can be assumed to 2). The case $k = 2$ is given alongside.




- Each occurrence of $x$ is represented by homes $h_{2i-1}$ and those of $\neg x$ by homes $h_{2i}$.
- Exactly two kinds of matchings possible:
  - "$T(x) = \textbf{true}$": $b_i$ with $g_i$ and $h_{2i}$.
  - "$T(x) = \textbf{false}$": $b_i$ with $g_{i-1}$ ($g_k$ if $i = 1$) and $h_{2i-1}$.

## Correctness of the reduction

- Note that "$T(x) = \textbf{true}$" matching leaves the homes for $x$ unoccupied and "$T(x) = \textbf{false}$" those for $\neg x$ unoccupied.
- For a clause $c$, the dedicated $b_c$ and $g_c$ are matched to a home that is left unoccupied when the variables are assigned truth values meaning that it corresponds to a **true** literal satisfying $c$.
- One more detail needs to be settled: there are more homes $H$ than boys $B$ and girls $G$ (but $|B| = |G|$).
- Add $l = |H| - |B|$ new boys and $l$ new girls. The $i$th such girl participates in $|H|$ triples with the $i$th boy and every home.
- Now a tripartite matching exists iff the set of clauses is satisfiable.





## Other problems involving sets

1. SET COVERING:
   INSTANCE: A family $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set $U$ and an integer $B$.
   QUESTION: Is there a set of $B$ sets in $F$ whose union is $U$?
2. SET PACKING:
   INSTANCE: A family $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set $U$ and an integer $K$.
   QUESTION: Is there a set of $K$ pairwise disjoint sets in $F$?
3. EXACT COVER BY 3-SETS:
   INSTANCE: A family $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set $U$ such that $|U| = 3m$ for some integer $m$ and for all $i$ $|S_i| = 3$.
   QUESTION: Is there a set of $m$ sets in $F$ that are disjoint and have $U$ as their union?





## Classifications obtained by generalization

**Corollary.** SET COVERING, SET PACKING, and EXACT COVER BY 3-SETS are all **NP**-complete.

- TRIPARTITE MATCHING can be reduced to EXACT COVER BY 3-SETS by noticing that it is a special case where $U$ is partitioned in three sets $B, G, H$ with $|B| = |G| = |H|$ and $F = \{\{b, g, h\} \mid (b, g, h) \in T\}$.
- EXACT COVER BY 3-SETS can be reduced to SET COVERING as a special case where the universe has $3m$ elements, all sets in $F$ have 3 elements and the budget $B = m$.
- EXACT COVER BY 3-SETS can be reduced to SET PACKING as a special case where the universe has $3m$ elements, all sets in $F$ have 3 elements and limit $K = m$. □





## A number problem: INTEGER PROGRAMMING

INSTANCE: a system of linear inequalities with integer coefficients.
QUESTION: Is there an integer solution of the system?

**Corollary.** INTEGER PROGRAMMING is **NP**-complete.

Proof. SET COVERING reducible to INTEGER PROGRAMMING:

Given a family $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set $U = \{u_1, \ldots, u_m\}$ and a integer $B$, construct a system:

$$\begin{array}{ll} 0 \le x_1 \le 1, \ldots, 0 \le x_n \le 1 & a_{11}x_1 + \cdots + a_{1n}x_n \ge 1 \\ \Sigma_{i=1}^{n} x_i \le B & \vdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \ge 1 \end{array}$$

where $a_{ij} = 1$ if $i$th element of $U$ is in the set $S_j$, otherwise $a_{ij} = 0$.
(The idea: $x_i = 1$ if $S_i$ in the cover and otherwise $x_i = 0$.) □

## Further problems involving numbers

1. LINEAR PROGRAMMING (i.e. INTEGER PROGRAMMING where non-integer solutions are allowed) is in **P**.
2. KNAPSACK:
   INSTANCE: A set of $n$ items with each item $i$ having a value $v_i$ and a weight $w_i$ (both positive integers) and integers $W$ and $K$.
   QUESTION: Is there a subset $S$ of the items such that $\Sigma_{i \in S} w_i \leq W$ but $\Sigma_{i \in S} v_i \geq K$?

**Theorem.** KNAPSACK is **NP**-complete.

Proof. We show that a simple special case of KNAPSACK is **NP**-complete where $v_i = w_i$ for all $i$ and $W = K$:

INSTANCE: A set of integers $w_1, \ldots, w_n$ and an integer $K$.
QUESTION: Is there a subset $S$ of the integers with $\Sigma_{i \in S} w_i = K$?





## Reduction from EXACT COVER BY 3-SETS

The reduction is based on the set $U = \{1, 2, \ldots, 3m\}$ and the sets $S_1, \ldots, S_n$ given as bit vectors $\{0,1\}^{3m}$ and $K = 2^{3m} - 1$. Then the task is to find a subset of bit vectors that sum to $K$.

$$\begin{array}{cccccc} \rightarrow & 0 & 1 & \ldots & 0 & 0 \\ & 1 & 0 & \ldots & 0 & 0 \\ & \vdots & & & & \\ \rightarrow & 0 & 0 & \ldots & 1 & 1 \\ \hline & 1 & 1 & \ldots & 1 & 1 \end{array}$$

- This does not quite work because of the carry bit, but the problem can be circumvented by using $n+1$ as the base rather than 2.
- Now $S_i$ corresponds to $w_i = \Sigma_{j \in S_i} (n+1)^{3m-j}$.
- Then a set of these integers $w_i$ adds up to $K = \Sigma_{j=0}^{3m-1} (n+1)^j$ iff there is an exact cover among $\{S_1, S_2, \ldots, S_n\}$. □





## 6. Pseudopolynomial Algorithms

**Proposition.** Any instance of KNAPSACK can be solved in $O(nW)$ time where $n$ is the number of items and $W$ is the weight limit.

Proof.

- Define $V(w,i)$: the largest value attainable be selecting some among the first $i$ items so that their total weight is exactly $w$.
- Each $V(w,i)$ with $w = 1, \ldots, W$ and $i = 1, \ldots, n$ can be computed by
$$V(w, i+1) = \max\{V(w,i), v_{i+1} + V(w - w_{i+1}, i)\}$$
where $V(w,0) = 0$ for all $w$ and $V(w,i) = -\infty$ if $w \leq 0$.
- For each entry this can be done in constant number of steps and there are $nW$ entries. Hence, the algorithm runs in $O(nW)$ time.
- An instance is answered "yes" iff there is an entry $V(w,i) \geq K$. □





## Strong NP-completeness

- The preceding algorithm is not polynomial w.r.t. the length of the input (which is $O(n \log W)$) but exponential ($W = 2^{\log W}$).
- An algorithm where the time bound is polynomial in the integers in the input (not their logarithms) is called *pseudopolynomial*.
- A problem is called **strongly NP-complete** if the problem remains **NP**-complete even if any instance of length $n$ is restricted to contain integers of size at most $p(n)$, for a polynomial $p$.
  ☞ Strongly **NP**-complete problems cannot have pseudopolynomial algorithms (unless $\mathbf{P} = \mathbf{NP}$).
- SAT, MAX CUT, TSP(D), HAMILTON PATH, … are strongly **NP**-complete but KNAPSACK is not.

## Yet another number problem: BIN PACKING

INSTANCE: $N$ positive integers $a_1, \ldots, a_N$ (items) and integers $C$ (capacity) and $B$ (number of bins).
QUESTION: Is there a partition of the numbers into $B$ subsets such that for each subset $S$, $\Sigma_{a_i \in S} a_i \leq C$?

- BIN PACKING is strongly **NP**-complete:
  Even if the integers are restricted to have polynomial values (w.r.t. the length of input), BIN PACKING remains **NP**-complete.
- Any pseudopolynomial algorithm for BIN PACKING would yield a polynomial algorithm for all problems in **NP** implying $\mathbf{P} = \mathbf{NP}$.





## Learning Objectives

- The concept of **NP**-completeness and its characterizations in terms of succinct certificates.
- You should know basic techniques to prove problems **NP**-complete and be able to construct such proofs on your own.
- A basic repertoire of **NP**-complete problems (related with satisfiability, graphs, sets, and numbers) to be used in further **NP**-completeness proofs.
- The definition of strong **NP**-completeness and awareness of number problems which are (not) strongly **NP**-complete.