Counting Problems

- ➤ Examples of counting problems
- ➤ The class #**P**
- ➤ Reductions and completeness
- ➤ The class $\oplus$**P**

(C. Papadimitriou: *Computational Complexity*, Chapter 18)

Counting Problems

- ➤ Previously we have considered two types of problems:
  decision problems (whether a solution exists)
  function (search) problems (find a solution)
- ➤ Now we consider a new type of a *counting problem* asking *how many solutions exist*.
- ➤ #SAT: given a Boolean expression, compute the number of different truth assignments that satisfy it.
- ➤ #HAMILTON PATH: compute the number of different Hamilton paths in a given graph.
- ➤ These are counting versions of **NP**-complete decision problems.

Counting problems—cont'd

- ➤ Counting the number of solutions can be highly nontrivial even if the decision problem is polynomial.
- ➤ An example is the problem of counting the number of perfect matchings of a bipartite graph.
- ➤ This corresponds to the problem of computing the *permanent* of a matrix

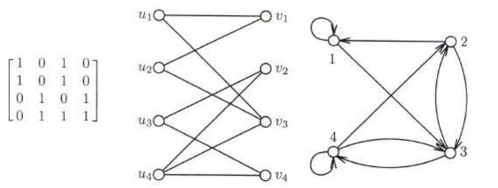
$$\mathrm{perm}\, A^G = \sum_{\pi} \prod_{i=1}^{n} A^G_{i,\pi(i)}$$

  where A^G is the adjacency matrix of the graph.
- ➤ This is why the problem is often called PERMANENT.

Counting problems—cont'd

- ➤ A bipartite graph with n "boys" $\{u_1,\ldots,u_n\}$ and n "girls" $\{v_1,\ldots,v_n\}$ can equivalently seen as a directed graph with nodes $\{1,\ldots,n\}$ where (i,j) is an edge in G' iff $[u_i,v_j]$ is an edge in G.
- ➤ Now a perfect matching corresponds to a *cycle cover*: a set of node-disjoint cycles that together cover all the nodes.

Example. [Papadimitriou, 1994]



For instance, a perfect matching $\{[u_1,v_3],[u_3,v_2],[u_2,v_1],[u_4,v_4]\}$ corresponds to a cycle cover $\{(1,3,2,1),(4,4)\}$.

Counting problems—cont'd

- Counting solutions is relevant, e.g., to probabilistic calculations.
- GRAPH RELIABILITY: count the number of subgraphs of a graph that contain a path from 1 to n.

  This number (divided by the number of subgraphs) gives the reliability of the graph: the probability that two nodes remain connected if all edges fail independently with probability $\frac{1}{2}$.

The class #P

- Let Q be a polynomially balanced and polynomial-time decidable binary relation. The *counting problem* associated with Q is the following: Given x, how many y are there such that $(x,y) \in Q$ (the answer given as a binary integer).

  The class #**P** is the class of all counting problems associated with polynomially balanced and polynomial-time decidable binary relations.
- For #SAT relation Q: $(x,y) \in Q$ iff a truth assignment y satisfies a Boolean expression x.
- For #HAMILTON PATH relation Q: $(x,y) \in Q$ iff y is a Hamilton path of a graph x.

#P-Completeness

- Counting problems can be ordered using *parsimonious reductions*.
- A parsimonious reduction from a counting problem A to a counting problem B is a function R which maps an instance x of A to an instance $R(x)$ of B such that the number of solutions of $R(x)$ is the same as that of x.
- Most reductions between **NP**-complete problems presented previously are parsimonious.
- A counting problem in #**P** is #**P**-complete if every problem in #**P** can be reduced to it with a parsimonious reduction.

The class #P—cont'd

Theorem. #SAT is #**P**-complete

Proof. Given $A \in$ #**P** with relation Q there is a poly-time TM M deciding Q. We can build a circuit $C(x)$ with $|x|^k$ inputs s.t. with input y output of $C(x)$ is true iff M accepts $x;y$ (Cook's theorem).

This is a parsimonious reduction to #CIRCUIT SAT which reduces to #SAT parsimoniously. (Parsimonious reductions compose.) □

- This implies directly that many counting versions of **NP**-complete problems are #**P**-complete.
- #HAMILTON PATH is #**P**-complete.

The class #P—cont'd

- Note: a polynomial algorithm for a search problem *does not* imply that the corresponding counting problem is solvable in polynomial time.
- A classical example is PERMANENT
- The corresponding search problem (finding a perfect matching of a bipartite graph) is solvable in polynomial time.
- However, PERMANENT is #**P**-complete.
- Notice that this implies that, for example, #SAT can be reduced to PERMANENT with a parsimonious reduction.

  (Hence, the reduction has to be complicated and indirect!)

The class #P—cont'd

- Notice that #**P** problems can be solved in polynomial space.
- How do **PH** and #**P** relate?
  (Remember: $\mathbf{PH} \subseteq \mathbf{PSPACE}$).
- Counting is stronger than the polynomial hierarchy!
- *Toda's theorem*: $\mathbf{PH} \subseteq \mathbf{P}^{\mathbf{PP}}$

  where **PP** effectively tells only whether the *first bit* of the number of accepting computations is zero or one.

THE CLASS $\oplus$P

- What about deciding the *last bit* of the number of accepting computations?
- $\oplus$SAT: Given a set of clauses, is the number of satisfying truth assignments odd?
- $L \in \oplus\mathbf{P}$ if there is a nondeterministic Turing machine N such that for all strings x, $x \in L$ iff the number of accepting computations of N on x is odd (or equivalently)
- $L \in \oplus\mathbf{P}$ if there is a polynomially balanced and polynomially decidable relation R such that $x \in L$ iff the number of ys such that $(x,y) \in R$ is odd.

$\oplus$P—cont'd

Theorem. $\oplus$SAT and $\oplus$HAMILTON PATH are $\oplus\mathbf{P}$-complete.

Theorem. $\oplus\mathbf{P}$ is closed under complement.

Proof. The complement of $\oplus$SAT (deciding whether the number of satisfying assignments is even) is $\mathbf{co}\oplus\mathbf{P}$-complete. We show that this problem reduces to $\oplus$SAT making $\oplus$SAT $\mathbf{co}\oplus\mathbf{P}$-complete. As $\oplus$SAT is also $\oplus\mathbf{P}$-complete, $\oplus\mathbf{P} = \mathbf{co}\oplus\mathbf{P}$ (the classes are closed under reductions).

Reducing the complement of $\oplus$SAT to $\oplus$SAT: Given a set of clauses on variables $x_1, \ldots, x_n$, (i) add the new variable z to each clause and (ii) add n clause $\neg z \vee x_i, i = 1, \ldots, n$. Now the number of satisfying truth assignment has increased by one, in which each variable true. $\square$

⊕P—cont'd

- $\oplus\mathbf{P}$ seems weaker than $\mathbf{PP}$: ⊕MATCHING is in $\mathbf{P}$.
- But not powerless:

Theorem. $\mathbf{NP} \subseteq \mathbf{RP}^{\oplus\mathbf{P}}$

Proof sketch.

- The idea is to show how an $\mathbf{NP}$-complete problem (SAT) can be solved using a Monte Carlo algorithm which uses ⊕SAT as its oracle.
- For the algorithm we define for a set of Boolean variables $S \subseteq \{x_1, \ldots, x_n\}$ a Boolean expression η_S stating that an even number among the variables in S are true as follows:

  Let $y_0, \ldots, y_n$ be new variables. Now η_S is the conjunction of the expressions $(y_0), (y_n)$, and for all $i = 1, \ldots, n$,
  $(y_i \leftrightarrow (y_{i-i} \oplus x_i))$, if $x_i \in S$ and $(y_i \leftrightarrow y_{i-i})$, if $x_i \notin S$.

Proof—cont'd

- The basic idea is that if we continue to add the requirement that an even number of variables are true in a random subset of the variables for n subsets, then with a reasonable probability one of the resulting expressions has a single satisfying truth assignment (which can be detected by the ⊕SAT oracle.
- Now an Monte Carlo algorithm for SAT using ⊕SAT as its oracle works as follows:

  Let ϕ_0 be the given expression ϕ.
  For $i = 1, \ldots, n+1$, repeat the following:
  Generate a random subset S_i of the variables and set
  $\phi_i = \phi_{i-1} \wedge \eta_{S_i}$.
  If $\phi_i \in$ ⊕SAT, then answer "ϕ is satisfiable".
  If after $n+1$ steps none of the ϕ_is is in ⊕SAT, then answer "ϕ is probably unsatisfiable".

Proof—cont'd

- Clearly, the algorithm does not have any false positives
- It can be shown that the probability of a false negative is no larger than 7/8.
- Hence, by repeating the algorithm six times the probability of a false negative is less than half. □

Learning Objectives

- The concept of counting problems.
- Classes $\#\mathbf{P}$ and $\oplus\mathbf{P}$.
- Parsimonious reductions and completeness
- Typical complete problems for $\#\mathbf{P}$ and $\oplus\mathbf{P}$.
- The relationship of $\#\mathbf{P}$ and $\oplus\mathbf{P}$ to other complexity classes.