On P vs NP

- The map of **NP**
- Isomorphism
- Density
- Oracle Turing machines
- Monotonic circuits

(C. Papadimitriou: *Computational Complexity*, Chapter 14)

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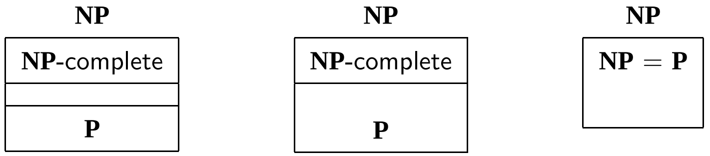
The Map of NP

- **NP**-completeness provides a powerful tool for classifying challenging computational problems.
- However, there are problems not known to be in **P** or **NP**-complete, such as GRAPH ISOMORPHISM.
- Are there problems in **NP** that are neither in **P** nor **NP**-complete, i.e., which is the case:



- The middle alternative is not possible.

Theorem. If $\mathbf{P} \neq \mathbf{NP}$, then there is a language in **NP** which is neither in **P** nor **NP**-complete.

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Isomorphism

- All known **NP**-complete languages are *polynomially isomorphic.*
- Languages K, L are polynomially isomorphic if there is a function $h : \Sigma^* \mapsto \Sigma^*$ such that
  (i) h is bijection;
  (ii) for each $x \in \Sigma^*$, $x \in K$ iff $h(x) \in L$;
  (iii) h and h^{-1} are polynomial-time computable.
- There is a polynomial-time mapping from any **NP**-complete problem to any other **NP**-complete problem (a logarithmic space reduction).
- However, a reduction is not necessarily a polynomial isomorphism (not a bijection) but with a *padding function* an isomorphism is obtained.

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Isomorphism—cont'd

- A function $pad : (\Sigma^*)^2 \mapsto \Sigma^*$ is a padding function for L if
  (i) pad is computable in logarithmic space;
  (ii) for each $x, y \in \Sigma^*$, $pad(x,y) \in L$ iff $x \in L$;
  (iii) for each $x, y \in \Sigma^*$, $|pad(x,y)| > |x| + |y|$;
  (iv) there is a logarithmic-space algorithm which given $pad(x,y)$ recovers y.

Example. A padding function for SAT:

Given x (a conjunction of m clauses with n variables) and a binary string y, $pad(x,y)$ is all clauses of x together with $m + |y|$ new clauses and $|y| + 1$ more variables where the m new clauses are copies of the clause u_{n+1} and where the $m+i$th new clause is either $\neg u_{n+i+1}$ or u_{n+i+1} depending on whether the ith symbol in y is 0 or 1.

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Isomorphism—cont'd

- If R is a reduction from K to L and pad is a padding function for L, $pad(R(x),x)$ is a length-increasing one-to-one reduction.

  Furthermore, there is a logarithmic-space algorithm for the inverse of $pad(R(x),x)$.
- If there is a reduction from K to L and a reduction from L to K and the reductions are length-increasing, one-to-one and logarithmic-space invertible, then K, L are polynomially isomorphic.
- Corollary: The following **NP**-languages are polynomial isomorphic: SAT, HAMILTON PATH, CLIQUE, MAX CUT, TRIPARTITE MATCHING, KNAPSACK, ...

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Density

Density of a language L:

$$\mathrm{dens}_L(n) = |\{x \in L : |x| \leq n\}|$$

Proposition. If $K, L \subseteq \Sigma^*$ are polynomially isomorphic, then dens_K and dens_L are polynomially related ($\mathrm{dens}_L(n) \leq \mathrm{dens}_K(p(n))$).

- A *sparse* language: polynomially bounded density function
- A *dense* language: superpolynomial density function

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Density—cont'd

- Examples:

  Any unary language ($\subseteq \{0\}^*$) is sparse ($\mathrm{dens}_L(n) \leq n$).

  All **NP**-complete language seen this far are dense.

Theorem. If a unary language $U \subseteq \{0\}^*$ is **NP**-complete, then $\mathbf{P} = \mathbf{NP}$.

Theorem. If a sparse language is **NP**-complete, then $\mathbf{P} = \mathbf{NP}$.

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Oracle Turing Machines

- The idea: study complexity in a setting where a part of the computation comes "for free".
- Can be used for exploring conjectures (like $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$) in an alternative setting.
- Can isolate orthogonal (independent) sources of complexity.

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Oracles—cont'd

- An *oracle Turing machine* $M^?$:
  - New elements: query string, query state $q?$, answer states q_{YES}, q_{NO}
  - From the query state $q?$ the machine moves to q_{YES} or to q_{NO} depending on whether $y \in A$ holds or not where y is the content of the query string and A the oracle set.
  - Note that *a query* is performed *in one step*!
  - Computation of $M^?$ with oracle A on input x: $M^A(x)$.
- For any time complexity class $\mathcal{C}$ and oracle A there is a corresponding complexity class $\mathcal{C}^A$.

Theorem. There is an oracle A for which $\mathbf{P}^A = \mathbf{NP}^A$.

Proof: Let A be **PSPACE**-complete. Then

$$\mathbf{PSPACE} \subseteq \mathbf{P}^A \subseteq \mathbf{NP}^A \subseteq \mathbf{NPSPACE} \subseteq \mathbf{PSPACE}$$

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Oracles—cont'd

Theorem. There is an oracle B for which $\mathbf{P}^B \neq \mathbf{NP}^B$.

Proof. B is constructed by "diagonalization".

Lessons to be learned w.r.t. $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$

- By the above $\mathbf{P} \neq \mathbf{NP}$ is possible; non-trivial conjecture.
- "Ordinary proof techniques" are not sufficient for establishing $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$.

  They are not affected by oracles.

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Monotone Circuits

- **Conjecture B.** **NP**-complete problems have no polynomial circuits, uniform or not.

  ☞ This implies $\mathbf{P} \neq \mathbf{NP}$ by Theorem 11.5 (A language L has uniformly polynomial circuits iff $L \in \mathbf{P}$).
- Lower bounds on the size of circuits for families of functions are hard to establish.
- Consider a weaker circuit model: monotone circuits (ones without NOT gates).
- Monotone circuits can only compute monotone functions: output cannot changed from **true** to **false** by changing input from **false** to **true**.

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Monotone Circuits—cont'd

- Some **NP**-complete problems are monotonic, e.g., HAMILTON PATH and CLIQUE.
- How small can the monotone circuits coding these problems be?
- Let $\text{CLIQUE}_{n,k}$ be the Boolean function deciding whether a graph $G = (V, E)$ with n nodes has a clique of size k.

Theorem. (Razborov's Theorem): There is a constant c such that for large enough n all monotone circuits for $\text{CLIQUE}_{n,k}$ with $k = n^{1/4}$ have size at least $2^{cn^{1/8}}$.

- To show $\mathbf{P} \neq \mathbf{NP}$ it would be sufficient to establish that all monotone languages in $\mathbf{P}$ have polynomial monotone circuits.
- However, this does not hold (e.g., for MATCHING).

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Learning Objectives

- The concepts of polynomial isomorphism and density of a language
- The concept of an oracle Turing machine
- The role of monotonic circuits in studying the $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ question.

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