# coNP AND FUNCTION PROBLEMS

- The class of **coNP**
- The relationship of **coNP** and **NP**
- The class **coNP** $\cap$ **NP**
- Function problems vs. decision problems
- Classes of function problems
- Total functions

(C. Papadimitriou: *Computational Complexity*, Chapter 10)

© 2007 TKK, Laboratory for Theoretical Computer Science



## 1. The class of complement problems coNP

- **NP** is the class of problems with succinct certificates.
- **coNP** $= \{L \mid \overline{L} \in \mathbf{NP}\}$ is the class of problems with succinct disqualifications.

  **Example.** Consider the problem of VALIDITY:
  INSTANCE: A Boolean expression $\phi$ in CNF.
  QUESTION: Is $\phi$ valid?
- VALIDITY is in **coNP**: for an expression $\phi$ which is not valid, a falsifying truth assignment is a succinct disqualification.
- HAMILTON PATH COMPLEMENT and SAT COMPLEMENT are also in **coNP**.
- **P** $\subseteq$ **coNP**

© 2007 TKK, Laboratory for Theoretical Computer Science



### coNP-completeness

**Definition.** A language $L$ is **coNP**-*complete* iff $L \in$ **coNP** and $L' \leq_{\mathrm{L}} L$ holds for every language $L' \in$ **coNP**.

**Proposition.** HAMILTON PATH COMPLEMENT and VALIDITY are **coNP**-complete.

Proof. Every language $L \in$ **coNP** is reducible to VALIDITY, because $\overline{L} \in$ **NP** and, hence, there is a reduction $R$ from $\overline{L}$ to SAT such that for every string $x$, $x \in \overline{L}$ iff $R(x) \in$ SAT. But then for a reduction $R'(x) = \neg R(x)$, $x \in L$ iff $R(x) \notin$ SAT iff $R'(x) = \neg R(x) \in$ VALIDITY.

Similarly for HAMILTON PATH COMPLEMENT. $\square$

© 2007 TKK, Laboratory for Theoretical Computer Science



## 2. The Relationship of coNP and NP

**Proposition.** If $L \subset \Sigma^*$ is **NP**-complete, then its complement $\overline{L} = \Sigma^* - L$ is **coNP**-complete.

Further observations:

- It is open whether **NP** $=$ **coNP**.
- If **P** $=$ **NP**, then **NP** $=$ **coNP** (and **P** $=$ **coNP**).
- It is possible that **P** $\neq$ **NP** but **NP** $=$ **coNP** (however, it is strongly believed that **NP** $\neq$ **coNP**).
- The problems in **coNP** that are **coNP**-complete are the least likely problems to be in **P** and also in **NP** (see below).

© 2007 TKK, Laboratory for Theoretical Computer Science

## Do coNP and NP coincide?

**Proposition.** If a **coNP**-complete problem is in **NP**, **NP** = **coNP**.

Proof.

Suppose that $L$ is a **coNP**-complete problem that is in **NP**.

($\supseteq$) Consider $L' \in$ **coNP**. Then there is a reduction $R$ from $L'$ to $L$. Then $L' \in$ **NP**, because $L'$ can be decided by a polynomial time NTM which on input $x$ computes first $R(x)$ and then starts the NTM for $L$.

($\subseteq$) Consider $L' \in$ **NP**. Then $\overline{L'} \in$ **coNP** and there is a reduction $R$ from $\overline{L'}$ to $L$. Then similarly $\overline{L'} \in$ **NP** and hence $L' \in$ **coNP**. $\square$

© 2007 TKK, Laboratory for Theoretical Computer Science



## The primality problem PRIMES

INSTANCE: An integer $N$ in binary representation.
QUESTION: Is $N$ a prime number?

- PRIMES $\in$ **coNP** as any divisor acts as a succinct disqualification.
- Note that a $O(\sqrt{N})$ algorithm for PRIMES testing all relevant divisor candidates is only pseudopolynomial.
- PRIMES $\in$ **NP** (as shown below) and hence PRIMES $\in$ **coNP** $\cap$ **NP**.
- New result in August 2002:
  M. Agrawal, N. Kayal, N. Saxena: *PRIMES is in* **P** !!

© 2007 TKK, Laboratory for Theoretical Computer Science



## 3. The Class coNP $\cap$ NP

- Problems in **coNP** $\cap$ **NP** have both succinct certificates and disqualifications.
- **P** $\subseteq$ **coNP** $\cap$ **NP** as **P** $\subseteq$ **coNP** and **P** $\subseteq$ **NP**.
- If two problems in **NP** are *dual*, i.e. each is *reducible to the complement* of the other, then both are in **coNP** $\cap$ **NP**.

**Example.**
MAX FLOW(D): Has a network $N$ a flow of at least $K$ from $s$ to $t$?
MIN CUT(D): Given a network, is there a set of edges of capacity of at most $B$ such that deleting these disconnects $s$ from $t$?

Now by the max flow–min cut theorem, $N$ *has* a flow of value at least $K$ iff it *does not have* a cut of capacity $K-1$ or less.

© 2007 TKK, Laboratory for Theoretical Computer Science



## PRIMES has succinct certificates

A succinct certificate for primality can be obtained using the following theorem.

**Theorem.** A number $p > 1$ is prime iff there is a number $1 < r < p$ such that $r^{p-1} = 1 \bmod p$ and, furthermore, $r^{\frac{p-1}{q}} \neq 1 \bmod p$ for all prime divisors $q$ of $p-1$.

**Corollary.** PRIMES is in **NP** $\cap$ **coNP**.

- The theorem provides a succinct certificate for the primality of $p$:
  $$C(p) = (r; q_1, C(q_1), \ldots, q_k, C(q_k))$$
  where $C(q_i)$ is a *recursive* primality certificate for each prime divisor $q_i$ of $p-1$.
- The recursion stops for prime divisors $q_i = 2$ for which $C(q_i) = (1)$.

© 2007 TKK, Laboratory for Theoretical Computer Science

## Verifying the certificate $C(p)$

The following observations can be made:

- The certificate $C(p)$ is polynomial in the length of $p$ (in $\log p$) and it can be checked by division and exponentiation.
- Ordinary multiplication and division are doable in polynomial time in the length of the input (in binary representation).
- Exponentiation $r^{p-1} \bmod p$ can be done in polynomial time by repeated squaring $r^1, r^2, r^4, \ldots r^{2^l} \pmod p$ where $l = \lfloor \log_2(p-1) \rfloor$ and then with at most $l$ additional multiplications.

☞ The certificate $C(p)$ can be checked in polynomial time. □

© 2007 TKK, Laboratory for Theoretical Computer Science



# 4. Function Problems vs. Decision Problems

- We have studied decision problems but many problems in practice require a more complicated answer than "yes"/"no".

  **Example.** Find a satisfying truth assignment for a formula.

  **Example.** Compute an optimal tour for TSP.
- Such problems are called *function problems*.
- Decision problems are useful surrogates of function problems only in the context of *negative complexity results*.

  **Example.** SAT and TSP(D) are **NP**-complete. Then unless $\mathbf{P} = \mathbf{NP}$, there is no polynomial time algorithm for finding a satisfying truth assignment or an optimal tour.

© 2007 TKK, Laboratory for Theoretical Computer Science



## The relationship of SAT and FSAT

FSAT: given a Boolean expression $\phi$, if $\phi$ is satisfiable then return a satisfying truth assignment of $\phi$ otherwise return "no".

- If FSAT can solved in polynomial time, then clearly so can SAT.
- If SAT can be solved in polynomial time, then so can FSAT using the following algorithm given input $\phi$ with variables $x_1, \ldots, x_n$ ($\phi[x = \mathbf{true}]$ denotes $\phi$ where variable $x$ is replaced by **true**):

  if $\phi \notin$ SAT then return "no";
  for all $x \in \{x_1, \ldots, x_n\}$ do
  if $\phi[x = \mathbf{true}] \in$ SAT then $T(x) := \mathbf{true}$; $\phi := \phi[x = \mathbf{true}]$
  else $T(x) := \mathbf{false}$ ; $\phi := \phi[x = \mathbf{false}]$;
  return $T$;

© 2007 TKK, Laboratory for Theoretical Computer Science



## The relationship of TSP(D) and TSP

- If TSP can solved in polynomial time, then clearly so can TSP(D).
- If TSP(D) can solved in polynomial time, then so can TSP in the following way.
- An optimal tour can be found using the algorithm below which finds
  1. the cost $0 \le C \le 2^n$ of an optimal tour by binary search and
  2. an optimal tour using the cost $C$ computed in step 1.

  (Here $n$ is the length of the encoding of the problem instance.)
- Both steps involve a polynomial number of calls to the polynomial time algorithm for TSP(D) (assuming that such an algorithm exists).

© 2007 TKK, Laboratory for Theoretical Computer Science

## An algorithm for TSP

An algorithm for TSP(D) is used as a subroutine:

/* Find the cost $C$ of an optimal tour by binary search*/
$C := 0$; $C_u := 2^n$;
while $(C_u > C)$ do
  if there is a tour of cost $\lfloor (C_u + C)/2 \rfloor$ or less then
    $C_u := \lfloor (C_u + C)/2 \rfloor$
  else $C := \lfloor (C_u + C)/2 \rfloor + 1$;
/* Find an optimal tour */
For all intercity distances do
  set the distance to $C + 1$;
  if there is a tour of cost $C$ or less, freeze the distance to $C + 1$
  else restore the original distance and add it to the tour;
endfor

© 2007 TKK, Laboratory for Theoretical Computer Science



## 5. Classes of Function Problems

**Definition.** Let $L \in \mathbf{NP}$. Then there is a polynomial time decidable and polynomially balanced relation $R_L$ such that for all strings $x$, there is a string $y$ with $R_L(x,y)$ iff $x \in L$.

The *function problem* associated with $L$ (denoted F$L$) is:

Given $x$, find a string $y$ such that $R_L(x,y)$ if such a string $y$ exists; otherwise return "no".

- The class of all function problems associated as above with languages in **NP** is called **FNP**.
- **FP** is the subclass of **FNP** solvable in polynomial time.
- FSAT is in **FNP** and FHORNSAT is in **FP** (but it is open whether TSP is in **FNP**).

© 2007 TKK, Laboratory for Theoretical Computer Science



## Reductions and completeness for function problems

A function problem $A$ *reduces* to a function problem $B$ if there are string functions $R$, $S$ computable in logarithmic space such that for all strings $x, z$: if $x$ is an instance of $A$, then $R(x)$ is an instance of $B$ and if $z$ is a correct output of $R(x)$, then $S(z)$ is a correct output of $x$.

- Reductions compose among function problems.
- A problem $A$ is *complete* for a class F$\mathcal{C}$ of function problems if it is in F$\mathcal{C}$ and every problem in F$\mathcal{C}$ reduces to $A$.
- **FP** and **FNP** are closed under reductions.
- FSAT is **FNP**-complete.
- $\mathbf{FP} = \mathbf{FNP}$ iff $\mathbf{P} = \mathbf{NP}$.

© 2007 TKK, Laboratory for Theoretical Computer Science



## 6. Total Functions

- There are certain important problems in **FNP** that are guaranteed to never return "no".

  **Example.** FACTORING: Given an integer $N$, find its prime decomposition $N = p_1^{k_1} \cdots p_m^{k_m}$.

  (No known polynomial time algorithm).
- FACTORING seems to be different from the other hard problems in **FNP**: it is a total function in a sense:

  **Definition.** A problem $L$ in **FNP** is called *total* if for every string $x$ there is at least one string $y$ such that $R_L(x,y)$.
- The subclass of **FNP** containing all total function problems is denoted by **TFNP**.

© 2007 TKK, Laboratory for Theoretical Computer Science

## Total functions—cont'd

There are also other problems in **TFNP** with no known polynomial time algorithm.

**Example.** HAPPYNET:
INSTANCE: An undirected graph $G = (V, E)$ with integer weights $w$ on edges.

GOAL: Find a state of the graph where all nodes are happy.

- A state is a mapping $S : V \longrightarrow \{-1, +1\}$.
- A node $i$ is happy in a state $S$ of $G = (V, E)$ if

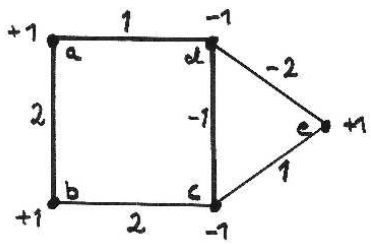
$$S(i) \cdot \sum_{[i,j] \in E} S(j) w[i, j] \geq 0.$$

© 2007 TKK, Laboratory for Theoretical Computer Science



**Example.**

- Consider the graph below and a state $S$ such that $S(a) = S(b) = S(e) = 1$ and $S(c) = S(d) = -1$
- Node $a$ is happy as $1 \cdot (1 \cdot 2 + -1 \cdot 1) = 1 \geq 0$
- Node $c$ is unhappy as $-1 \cdot (1 \cdot 2 + -1 \cdot -1 + 1 \cdot 1) = -2 < 0$




© 2007 TKK, Laboratory for Theoretical Computer Science



## Properties of HAPPYNET

- Every instance is guaranteed to have a happy state which can be found using the following algorithm:

  Start with any $S$ and while there is an unhappy node, flip it.
- This algorithm is not polynomial but pseudopolynomial $O(W)$ where $W$ is the sum of all weights.
- No polynomial algorithm known.
- HAPPYNET is equivalent with finding stable states in neural networks in the Hopfield model.

© 2007 TKK, Laboratory for Theoretical Computer Science



## Other total functions

- ANOTHER HAMILTON CYCLE is **FNP**-complete.
- ANOTHER HAMILTON CYCLE for cubic graphs is in **TFNP**.
- EQUAL SUMS:

  Given $n$ positive integers $a_1, \ldots, a_n$ such that $\sum_{i=1}^{n} a_i < 2^n - 1$, find two different subsets that have the same sum.
- EQUAL SUMS in **TFNP**.

  Proof. There are $2^n$ subsets of $a_1, \ldots, a_n$ and for each of them the sum is an integer between $0$ and $2^n - 2$.

  Assume that all subsets have different sums. Then there are $2^n$ different integers between $0$ and $2^n - 2$, a contradiction. Hence, there are two different subsets that have the same sum. □

© 2007 TKK, Laboratory for Theoretical Computer Science

## Learning Objectives

- The definition of **coNP** and examples of languages from this class, e.g., VALIDITY.
- The characterization of **coNP** based on disqualifications.
- Reductions and completeness for function problems
- Relationship of decision problems and function problems

© 2007 TKK, Laboratory for Theoretical Computer Science