14.5.9

A language $L \subseteq \{0,1\}^*$ has *polynomial circuits* if there is a family $\mathcal{C} = (C_0, C_1, \ldots)$ such that (i) the size of C_n is at most $p(n)$ for some fixed polynomial p and (ii) $x = x_1 x_2 \ldots x_n \in L$ iff the output of C_n is true when the ith input variable is true if $x_i = 1$ and false otherwise.

A language A is *sparse* if its density function $\text{dens}_A(n) = |\{x \in A : |x| \leq n\}|$ is bounded by a fixed polynomial.

We shall show that a language L has polynomial circuits iff $L \in \mathbf{P}^A$ for some sparse language A.

The "if" direction. Assume that $L \in \mathbf{P}^A$ for some sparse language A such that A's density function is bounded by a polynomial $Q(n)$. Without loss of generality, assume that $A \subseteq \{0,1\}^*$. Let M be a deterministic machine that decides L in polynomial time using the oracle for A. Let the polynomial time bound function be $P(n)$. Now the maximum length of the oracle tape in any computation on input x is at most $P(|x|)$. Because A's density function is bounded by $Q(n)$, then the maximum number of strings in A of length $P(|x|)$ is at most $Q(P(|x|))$, a polynomial.

For each n there is a polynomial circuit $C_{|x|}$ such that $x = x_1 x_2 \ldots x_n \in L$ iff the output of the circuit is true for x. The circuit is like in the Cook's theorem (Theorem 8.2) except that we augment the circuit with "look-up" sub-circuits with $P(|x|)$ inputs deciding whether the contents on the oracle tape belong to A (the "look-up" sub-circuit is connected to the variables describing the current state of the oracle tape). Clearly the size of the "look-up" sub-circuit is $\mathcal{O}(P(|x|) \cdot Q(P(|x|)))$ (a disjunction of all codings of strings of length $P(|x|)$ belonging to A). Therefore the resulting circuit is polynomial and L has polynomial circuits if $L \in \mathbf{P}^A$ for some sparse language A.

The "only if" direction. Let L have polynomial circuits $\mathcal{C} = (C_0, C_1, \ldots)$. Let $\zeta(C_i)$ be the coding of circuit C_i in an alphabet Σ. Clearly $|\zeta(C_i)|$ is of polynomial length w.r.t. the size of C_i and thus w.r.t. i for any reasonable coding.

Consider the language A over Σ such that $x; y \in A$ iff $x \in \{0\}^*$ and y is a prefix of $\zeta(C_{|x|})$. Clearly A is sparse because for each n, there are no strings of length at most n in A concerning circuits C_m, $m > n$, and furthermore each coding of a circuit C_i, $i \leq n$, has at most polynomial number of prefixes w.r.t. n.

Now there is the following $\mathbf{P}^A$-machine M deciding L. On input x, M first counts the length n of x. It then systematically tests strings $0^n; \sigma$ where $\sigma \in \Sigma$ by using the oracle for a A until it finds a σ' which is a prefix of $\zeta(C_n)$. It then proceeds to test strings of form $0^n; \sigma'\sigma$ and so on until it has recovered the coding of C_n. This is done in time proportional to $|\Sigma| \cdot |\zeta(C_n)|$ which is a polynomial w.r.t. n. It then evaluates C_n on x and says yes iff the output gate was true.

Therefore $L \in \mathbf{P}^A$ for some sparse language A if it has polynomial circuits.

15.5.6 (a)

Let $\mathcal{C} = (C_0, C_1, \ldots)$ be a uniform family of circuits and let $f(n)$ and $g(n)$ be functions from integers to integers. The *parallel time* of $\mathcal{C}$ is at most $f(n)$ if for all n the depth of C_n is at most $f(n)$. We say that the *total work* of $\mathcal{C}$ is at most $g(n)$ if for all n the size of C_n is at most $g(n)$.

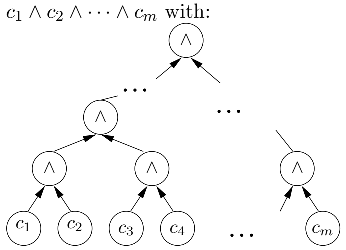
$\mathbf{NC}_j$ is the class of all languages $L \subseteq \{0,1\}^*$ such that there is uniform family of circuits $\mathcal{C}$ deciding L with $\mathcal{O}(\log^j n)$ parallel time and $\mathcal{O}(n^k)$ work.

The class $\mathbf{AC}_j$ is similar to $\mathbf{NC}$ except that the AND- and OR-gates in circuits can have unbounded fan-in.

We are to show that $\mathbf{NC}_i \subseteq \mathbf{AC}_i \subseteq \mathbf{NC}_{i+1}$.

The first inclusion holds trivially because each $\mathbf{NC}_i$ circuit is also an $\mathbf{AC}_i$ circuit.

Now take any AND- or OR-gate with fan-in $m > 2$. Replace this gate $c_1 \wedge c_2 \wedge \cdots \wedge c_m$ with:



Clearly the height of the replacing circuit is $\mathcal{O}(\log m)$ and the number of gates in it is $\mathcal{O}(m)$. Therefore, given an $\mathbf{AC}_i$ circuit C_n, we replace each gate with fan-in more than 2 with the equivalent set of gates with fan-in 2. The height of the resulting circuit is $\mathcal{O}(k \cdot \log n \cdot \log^i n)$ which equals to $\mathcal{O}(\log^{i+1} n)$ since the maximum fan-in (m) in C_n is at most the number n^k

of gates in C_n. On the other hand, the size of resulting circuit is at most $\mathcal{O}(n^k \cdot \max\{m\})$ where $\max\{m\}$ is the maximum fan-in in C_n which, as mentioned, is at most n^k — therefore the size of the resulting circuit is at most $\mathcal{O}(n^{2k})$, a polynomial. Thus the resulting circuit is a $\mathbf{NC}_{i+1}$ circuit. Since $\mathbf{AC}_i$ is uniform, there is a log-space algorithm generating the circuits in it. This same algorithm augmented with replacing the gates with fan-in more than 2 is clearly a log-space algorithm, too. Therefore, for each uniform family of $\mathbf{AC}_i$ circuits there is also a uniform family of $\mathbf{NC}_{i+1}$ circuits and the second inclusion holds.

From the inclusions $\mathbf{NC}_i \subseteq \mathbf{AC}_i \subseteq \mathbf{NC}_{i+1}$ is it easy to see that $\mathbf{NC} = \bigcup_i \mathbf{NC}_i = \bigcup_i \mathbf{AC}_i = \mathbf{AC}$.