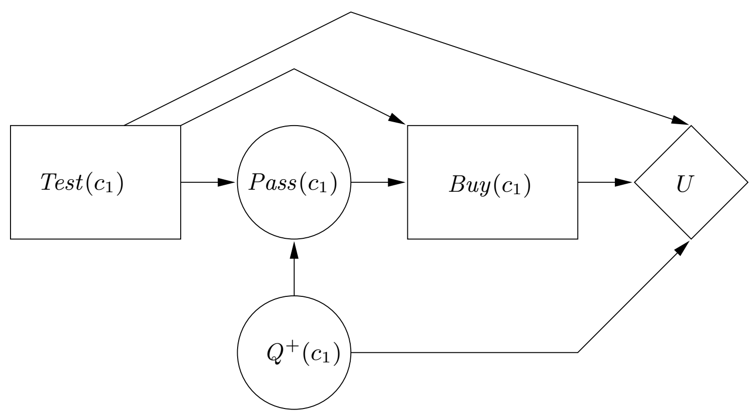
T-79.5102 **Autumn 2008**
Special Course in Computational Logic
Tutorial 7
Solutions

1. The following Boolean variables are introduced to model the domain

$Q^+(c_1)$:	The car c_1 is in good shape.
$Pass(c_1)$:	The car c_1 passed the test t_1.
$Test(c_1)$:	The buyer orders the test t_1 on c_1.
$Buy(c_1)$:	The buyer decides to buy c_1.

The last two are interpreted as pairs of actions: $\neg test(c_1)$ means that the buyer chooses to skip the test t_1 on c_1 whereas $\neg buy(c_1)$ means that the buyer decides not to buy c_1.

(a) The following decision network models the used-car buyer's dilemma:



However, this is out of scope in view of decision networks involving only one decision node. One possibility is to reduce the network given in the figure according to the two values of $Test(c_1)$. If it is false, then drop the nodes for $Test(c_1)$ and $Pass(c_1)$, and all arrows associated with them. Otherwise, drop $Test(c_1)$ and the respective arrows.

(b) Given $\neg test(c_1)$, the expected utilities for the buyer are

$$\begin{aligned}
\mathrm{EU}(buy(c_1)) &= P(q^+(c_1))\mathrm{U}(q^+(c_1)) + P(\neg q^+(c_1))\mathrm{U}(\neg q^+(c_1)) \\
&= 0.7 \times (2000€ - 1500€) + \\
&\quad\; 0.3 \times (2000€ - 1500€ - 700€) \\
&= 350€ - 60€ = 290€.
\end{aligned}$$

and $\mathrm{EU}(\neg buy(c_1)) = 0€$.

(c) Using Bayes' theorem, we obtain

$$
\begin{aligned}
& P(q^+(c_1) \mid pass(c_1)) \\
= \; & \beta P(pass(c_1) \mid q^+(c_1)) P(q^+(c_1)) \\
= \; & \beta \times 0.80 \times 0.70 = \beta \times 0.56 = \tfrac{0.56}{0.665} \approx 0.8421
\end{aligned}
$$

$$
\begin{aligned}
& P(\neg q^+(c_1) \mid pass(c_1)) \\
= \; & \beta P(pass(c_1) \mid \neg q^+(c_1)) P(\neg q^+(c_1)) \\
= \; & \beta \times 0.35 \times 0.30 = \beta \times 0.105 = \tfrac{0.105}{0.665} \approx 0.1579
\end{aligned}
$$

$$
\begin{aligned}
& P(q^+(c_1) \mid \neg pass(c_1)) \\
= \; & \gamma P(\neg pass(c_1) \mid q^+(c_1)) P(q^+(c_1)) \\
= \; & \gamma \times 0.20 \times 0.70 = \gamma \times 0.14 = \tfrac{0.14}{0.335} \approx 0.4179
\end{aligned}
$$

$$
\begin{aligned}
& P(\neg q^+(c_1) \mid \neg pass(c_1)) \\
= \; & \gamma P(\neg pass(c_1) \mid \neg q^+(c_1)) P(\neg q^+(c_1)) \\
= \; & \gamma \times 0.65 \times 0.30 = \gamma \times 0.195 = \tfrac{0.195}{0.335} \approx 0.5821
\end{aligned}
$$

where $\beta = \frac{1}{0.665}$ and $\gamma = \frac{1}{0.335}$ are normalizing constants so that $P(pass(c_1)) = \frac{1}{\beta} = 0.665$ and $P(\neg pass(c_1)) = \frac{1}{\beta} = 0.335$.

(d) The expected utilities are

$$
\begin{aligned}
& \mathrm{EU}(buy(c_1) \mid pass(c_1)) \\
= \; & P(q^+(c_1) \mid pass(c_1)) \mathrm{U}(q^+(c_1)) + \\
& P(\neg q^+(c_1) \mid pass(c_1)) \mathrm{U}(\neg q^+(c_1)) \\
= \; & 0.8421 \times 500€ + 0.1579 \times (-200€) \\
= \; & 421.05€ - 31.58€ = 389.47€.
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{EU}(buy(c_1) \mid \neg pass(c_1)) \\
= \; & P(q^+(c_1) \mid \neg pass(c_1)) \mathrm{U}(q^+(c_1)) + \\
& P(\neg q^+(c_1) \mid \neg pass(c_1)) \mathrm{U}(\neg q^+(c_1)) \\
= \; & 0.4179 \times 500€ + 0.5821 \times (-200€) \\
= \; & 208.95€ - 116.42€ = 92.53€
\end{aligned}
$$

whereas for not buying they are 0€.

(e) Since $buy(c_1)$ is the best action in any case, we obtain

$$
\begin{aligned}
& \mathrm{VPI}(Pass(c_1)) \\
= \; & P(pass(c_1)) \mathrm{EU}(buy(c_1) \mid pass(c_1)) + \\
& P(\neg pass(c_1)) \mathrm{EU}(buy(c_1) \mid \neg pass(c_1)) - \\
& \mathrm{EU}(buy(c_1)) \\
= \; & 0.665 \times 389.47€ + 0.335 \times 92.53€ - 290€ \approx 0€.
\end{aligned}
$$

Thus performing the test t_1 on c_1 does not bring any additional value for the buyer and in particular, given the price of the test (50€).

2. The goal is to prove in general that

$$
\mathrm{VPI}_E(X) = (\sum_x P(x \mid E) \mathrm{EU}(\alpha_x \mid E, x)) - \mathrm{EU}(\alpha \mid E) \geq 0 \qquad (1)
$$

where α is the best current action given E. Its utility is defined by

$$\mathrm{EU}(\alpha \mid E) = \max_A \mathrm{EU}(A \mid E) = \max_A \sum_i \mathrm{U}(R_i(A))P(R_i(A) \mid E)$$

and each α_x is similarly defined for evidence consisting of E and x.

$$\begin{aligned}
\mathrm{EU}(\alpha \mid E) &= \max_A \sum_i \mathrm{U}(R_i(A))P(R_i(A) \mid E) \\
&= \max_A \textstyle\sum_i \mathrm{U}(R_i(A)) \sum_x P(R_i(A) \mid E, x)P(x \mid E) \\
&= \max_A \textstyle\sum_i \sum_x P(x \mid E)\mathrm{U}(R_i(A))P(R_i(A) \mid E, x) \\
&= \max_A \textstyle\sum_x \sum_i P(x \mid E)\mathrm{U}(R_i(A))P(R_i(A) \mid E, x) \\
&= \max_A \textstyle\sum_x P(x \mid E) \sum_i \mathrm{U}(R_i(A))P(R_i(A) \mid E, x) \\
&\leq \textstyle\sum_x \max_A P(x \mid E) \sum_i \mathrm{U}(R_i(A))P(R_i(A) \mid E, x) \\
&= \textstyle\sum_x P(x \mid E) \max_A \sum_i \mathrm{U}(R_i(A))P(R_i(A) \mid E, x) \\
&= \textstyle\sum_x P(x \mid E)\mathrm{EU}(\alpha_x \mid E, x).
\end{aligned}$$

Now (1) follows by this inequality so that $\mathrm{VPI}_E(X) \geq 0$. Note that E can be replaced by a vector $\mathbf{E}$ if appropriate and hence $\mathrm{Do}(A)$ can be viewed as a part of $\mathbf{E}$ (c.f. R&N).