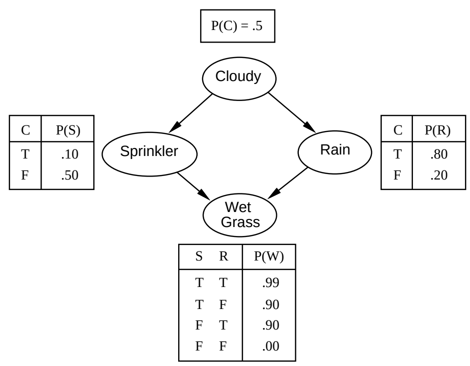
T-79.5102 **Autumn 2008**
Special Course in Computational Logic
Tutorial 5
Solutions

1. (a) There are four Boolean variables C, S, R, and WG. For two of them we have observed values s and wg while the values of the other two variables (non-observables) remain open. This leads to four possible atomic events as well as states for the respective Markov chain:

$$\mathbf{x}_1 = \{c, r\},\ \mathbf{x}_2 = \{c, \neg r\},\ \mathbf{x}_3 = \{\neg c, r\},\text{ and } \mathbf{x}_4 = \{\neg c, \neg r\}.$$

(b) Given the four states abobe, the respective transition matrix Q contains $4^2 = 16$ entries. The MCMC algorithm uses $\mathbf{P}(X_i \mid mb(X_i))$ to sample new values for non-observables independently of each other. Thus we need to determine $\mathbf{P}(C \mid mb(C))$ and $\mathbf{P}(R \mid mb(R))$. The Markov blankets in question are easily read off from the figure:

$$mb(C) = \{S, R\} \text{ and } mb(R) = \{C, S, WG\}.$$



The first sampling distribution is determined by

$$\begin{aligned} & \mathbf{P}(C \mid s, r) \\ = \ & \alpha \mathbf{P}(s \mid C)\mathbf{P}(r \mid C)\mathbf{P}(C) && \text{Eq. (14.11)} \\ = \ & \alpha\langle \tfrac{1}{10} \times \tfrac{4}{5} \times \tfrac{1}{2}, \tfrac{1}{2} \times \tfrac{1}{5} \times \tfrac{1}{2}\rangle \\ = \ & \alpha\langle \tfrac{4}{100}, \tfrac{5}{100}\rangle = \tfrac{100}{9}\langle \tfrac{4}{100}, \tfrac{1}{100}\rangle = \langle \tfrac{4}{9}, \tfrac{5}{9}\rangle \end{aligned}$$

and

$$\begin{aligned} & \mathbf{P}(C \mid s, \neg r) \\ = \ & \alpha \mathbf{P}(s \mid C)\mathbf{P}(\neg r \mid C)\mathbf{P}(C) && \text{Eq. (14.11)} \\ = \ & \alpha\langle \tfrac{1}{10} \times \tfrac{1}{5} \times \tfrac{1}{2}, \tfrac{1}{2} \times \tfrac{4}{5} \times \tfrac{1}{2}\rangle \\ = \ & \alpha\langle \tfrac{1}{100}, \tfrac{20}{100}\rangle = \tfrac{100}{21}\langle \tfrac{1}{100}, \tfrac{20}{100}\rangle = \langle \tfrac{1}{21}, \tfrac{20}{21}\rangle. \end{aligned}$$

Quite similarly, for the second sampling distribution, we have

$$\begin{aligned} & \mathbf{P}(R \mid c, s, wg) \\ = \ & \alpha \mathbf{P}(wg \mid s, R)\mathbf{P}(R \mid c) && \text{Eq. (14.11)} \\ = \ & \alpha\langle \tfrac{99}{100} \times \tfrac{4}{5}, \tfrac{90}{100} \times \tfrac{1}{5}\rangle \\ = \ & \alpha\langle \tfrac{396}{500}, \tfrac{90}{500}\rangle = \tfrac{500}{486}\langle \tfrac{396}{500}, \tfrac{90}{500}\rangle = \langle \tfrac{22}{27}, \tfrac{5}{27}\rangle \end{aligned}$$

and

$$
\begin{aligned}
& \mathbf{P}(R \mid \neg c, s, wg) \\
= \; & \alpha \mathbf{P}(wg \mid s, R) \mathbf{P}(R \mid \neg c) && \text{Eq. (14.11)} \\
= \; & \alpha \langle \tfrac{99}{100} \times \tfrac{1}{5}, \tfrac{90}{100} \times \tfrac{4}{5} \rangle \\
= \; & \alpha \langle \tfrac{99}{500}, \tfrac{360}{500} \rangle = \tfrac{500}{459} \langle \tfrac{99}{500}, \tfrac{360}{500} \rangle = \langle \tfrac{11}{51}, \tfrac{40}{51} \rangle.
\end{aligned}
$$

There are at least two approaches to carry out sampling. In the first one, the values C and R are sampled with equal probability, i.e., each sampling step is preceded by a choice between C and R. This model leads to transition probabilities summarized in the following table:

$q(\mathbf{x} \to \mathbf{x}')$	$\{c, r\}$	$\{c, \neg r\}$	$\{\neg c, r\}$	$\{\neg c, \neg r\}$
$\{c, r\}$	$\frac{1}{2}\times\frac{4}{9}+\frac{1}{2}\times\frac{22}{27}$	$\frac{1}{2}\times\frac{5}{27}$	$\frac{1}{2}\times\frac{5}{9}$	0
$\{c, \neg r\}$	$\frac{1}{2}\times\frac{22}{27}$	$\frac{1}{2}\times\frac{1}{21}+\frac{1}{2}\times\frac{5}{27}$	0	$\frac{1}{2}\times\frac{20}{21}$
$\{\neg c, r\}$	$\frac{1}{2}\cdot\frac{4}{9}$	0	$\frac{1}{2}\times\frac{5}{9}+\frac{1}{2}\times\frac{11}{51}$	$\frac{1}{2}\times\frac{40}{51}$
$\{\neg c, \neg r\}$	0	$\frac{1}{2}\times\frac{1}{21}$	$\frac{1}{2}\times\frac{11}{51}$	$\frac{1}{2}\times\frac{20}{21}+\frac{1}{2}\times\frac{40}{51}$

These transition probabilities form the transition matrix of the respective Markov chain:

$$
Q = \begin{bmatrix}
\frac{34}{54} & \frac{5}{54} & \frac{15}{54} & 0 \\
\frac{154}{378} & \frac{44}{378} & 0 & \frac{180}{378} \\
\frac{68}{306} & 0 & \frac{118}{306} & \frac{120}{306} \\
0 & \frac{17}{714} & \frac{77}{714} & \frac{620}{714}
\end{bmatrix}
$$

The other possibility is to sample C and R, e.g., in this order. This corresponds to the MCMC-ASK algorithm (on page 517) and transitions between states take place in two phases:

$q(\mathbf{x} \to \mathbf{x}')$	$\{c, r\}$	$\{c, \neg r\}$	$\{\neg c, r\}$	$\{\neg c, \neg r\}$
$\{c, r\}$	$\frac{4}{9}$	0	$\frac{5}{9}$	0
$\{c, \neg r\}$	0	$\frac{1}{21}$	0	$\frac{20}{21}$
$\{\neg c, r\}$	$\frac{4}{9}$	0	$\frac{5}{9}$	0
$\{\neg c, \neg r\}$	0	$\frac{1}{21}$	0	$\frac{20}{21}$

$q(\mathbf{x} \to \mathbf{x}')$	$\{c, r\}$	$\{c, \neg r\}$	$\{\neg c, r\}$	$\{\neg c, \neg r\}$
$\{c, r\}$	$\frac{22}{27}$	$\frac{5}{27}$	0	0
$\{c, \neg r\}$	$\frac{22}{27}$	$\frac{5}{27}$	0	0
$\{\neg c, r\}$	0	0	$\frac{11}{51}$	$\frac{40}{51}$
$\{\neg c, \neg r\}$	0	0	$\frac{11}{51}$	$\frac{40}{51}$

The alternative transition matrix Q' is obtained as a product of the respective transition matrices Q_C and Q_R.

(c) The matrix Q^2 represents transition probabilities in two steps.

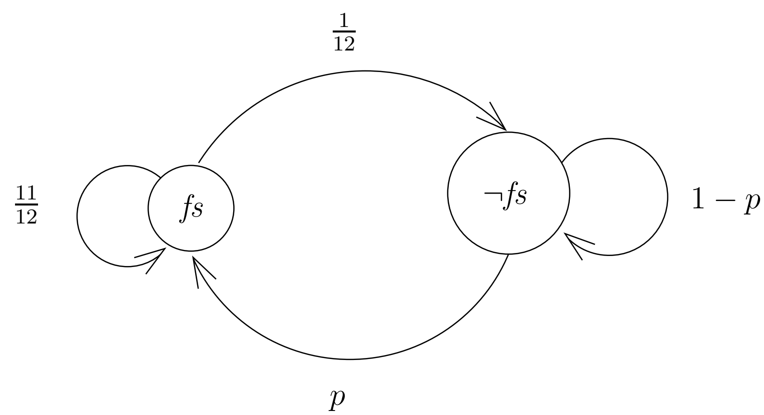
(d) Each row of the matrix Q^n approach the posterior probability distribution for the states of the chain. One strategy to compute Q^n is to compute repeated squares: $Q^2 = Q \times Q$, $Q^4 = Q^2 \times Q^2$, $Q^8 = Q^4 \times Q^4$, $\ldots$, Q^{2^k}, $\ldots$. For instance, every row in Q^{64} consists of the following probabilities: 0.1424, 0.0324, 0.1780, 0.6472. For Q' the convergence takes place much faster, i.e., $(Q')^{16}$ is enough.

To make a comparison, we may calculate exact values as follows:

$$
\begin{aligned}
\mathbf{P}(C, R \mid s, wg) &= \alpha \mathbf{P}(wg \mid s, R)\mathbf{P}(R \mid C)\mathbf{P}(s \mid C)\mathbf{P}(C) \\
&= \alpha\langle \tfrac{396}{5000}, \tfrac{90}{5000}, \tfrac{495}{5000}, \tfrac{1800}{5000}\rangle \\
&= \tfrac{5000}{2781}\langle \tfrac{396}{5000}, \tfrac{90}{5000}, \tfrac{495}{5000}, \tfrac{1800}{5000}\rangle \\
&= \langle \tfrac{396}{2781}, \tfrac{90}{2781}, \tfrac{495}{2781}, \tfrac{450}{2781}\rangle \\
&\approx \langle 0.1424, 0.0324, 0.1780, 0.6472\rangle.
\end{aligned}
$$

Note that $P(C)$ can be dropped from the expression in practise because the probability is always $\frac{1}{2}$ regardless of the value of C.

2. A fire station has one fire truck. Upon an emergency call, the truck goes out to fight fire and then returns to the station. Our representation of the domain is based on one Boolean variable FS which means that "the truck is at the fire station". The respective Hidden Markov Model (HMM) is illustrated in the figure; probabilities are assigned below in item (a).



(a) We assume that each day is divided into 24 time slices (one hour each). The probability of an alert is $\frac{1}{12}$. Further calculations are required to determine p, i.e., the returning probability of the truck after one time slice. The expected duration of one fire mission is

$$
\begin{aligned}
& p + (1-p) + p(1-p) + (1-p)^2 + p(1-p)^2 + (1-p)^3 + \ldots \\
= {} & 1 + (1-p) + (1-p)^2 + (1-p)^3 \ldots \\
= {} & \tfrac{1}{1-(1-p)} = \tfrac{1}{p}
\end{aligned}
$$

hours. From that we learn $p = \frac{1}{3}$.

(b) The transition matrix

$$
Q = \begin{bmatrix} \frac{11}{12} & \frac{1}{12} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}
$$

whereas the transition model is determined by the following CPT:

FS_t	$P(fs_{t+1})$	$P(\neg fs_{t+1})$
T	$\frac{11}{12}$	$\frac{1}{12}$
F	$\frac{1}{3}$	$\frac{2}{3}$

(c) To find out a stationary distribution $\mathbf{P}(FS_t) = \langle q, 1-q \rangle$ for sufficiently large t, we may view $\mathbf{P}(FS_t)$ as a message/vector

$$f = \begin{bmatrix} q \\ 1-q \end{bmatrix}$$

satisfying $f = Q^T f$ (forward reasoning without sensor model). By expanding the given product of matrices yiels two equations:

$$\left\{ \begin{array}{rcl} q & = & \frac{11}{12} \times q + \frac{1}{3} \times (1-q) \\ 1-q & = & \frac{1}{12} \times q + \frac{2}{3} \times (1-q) \end{array} \right. \tag{1}$$

Both equations turn out to be equivalent to $q = \frac{4}{5}$ which implies $\mathbf{P}(FS) = \langle \frac{4}{5}, \frac{1}{5} \rangle$ in the long run. Hence the expected time spent at the fire station is $\frac{4}{5}$ of 24 hours, i.e., 19 hours and 12 minutes.

(d) The goal is to determine an exact expression for $\mathbf{P}(FS_t)$ given an initial distribution $\mathbf{P}(FS_0) = \langle r, 1-r \rangle$. Basically, this is obtained by computing higher powers of the (transposed) transition matrix Q^T and multiplying $\langle r, 1-r \rangle$ with that. The result is

$$\mathbf{P}(FS_t) = \langle \tfrac{4}{5} + (r - \tfrac{4}{5})(\tfrac{7}{12})^t, \tfrac{1}{5} + ((1-r) - \tfrac{1}{5})(\tfrac{7}{12})^t \rangle$$

where $\frac{7}{12} = 1 - \frac{1}{12} - \frac{1}{3}$. Let us perform yet another prediction step to verify the correctness of the distribution. Using (1), we obtain

$$\begin{array}{rcl} P(fs_{t+1}) & = & \frac{11}{12}(\frac{4}{5} + (r - \frac{4}{5})(\frac{7}{12})^t) + \frac{1}{3}(\frac{1}{5} + ((1-r) - \frac{1}{5})(\frac{7}{12})^t) \\ & = & \frac{4}{5} + (\frac{11}{12}(r - \frac{4}{5}) + \frac{1}{3}((1-r) - \frac{1}{5}))(\frac{7}{12})^t \\ & = & \frac{4}{5} + (\frac{7}{12}r - \frac{7}{15})(\frac{7}{12})^t \\ & = & \frac{4}{5} + (r - \frac{4}{5})(\frac{7}{12})^{t+1}. \end{array}$$

A similar calculation shows that $P(\neg fs_{t+1}) = \frac{1}{5} + ((1-r) - \frac{1}{5})(\frac{7}{12})^{t+1}$.

(e) As to be expected, the limit $\lim_{t\to\infty} \mathbf{P}(FS_t) = \langle \frac{4}{5}, \frac{1}{5} \rangle$.