Making Simple Decisions

Outline
- Combining Beliefs and Desires Under Uncertainty
- The Basis of Utility Theory
- (Multiattribute) Utility Functions
- Decision Networks
- The Value of Information
- Decision-Theoretic Expert Systems

Based on the textbook by Stuart Russell & Peter Norvig:

Artificial Intelligence, A Modern Approach (2nd Edition)

Chapter 16

Maximum Expected Utility (MEU)

- The expected utility of an action \( A \) is
  \[
  EU(A | E) = \sum_i P(\text{Result}_i(A) | \text{Do}(A), E) \times U(\text{Result}_i(A)).
  \]
- The principle of maximum expected utility: a rational agent should choose an action that maximizes its expected utility.
- The MEU principle is closely related to performance measures: "If an agent maximizes a utility function that correctly reflects the performance measure by which its behavior is being judged, then it will achieve the highest possible performance score if averaged over the environments in which the agent could be placed."
- In this lecture, we concentrate on one-shot decisions. The case of making sequential decisions will be considered later.

1. Combining Beliefs and Desires

- A state \( S \) is a complete snapshot of the world.
- An agent’s preferences are captured by a utility function \( U \) which maps a state \( S \) to a number \( U(S) \) describing the desirability of \( S \).
- Specifying a utility function \( U \) for each state \( S \) may be tedious.
- The problem can be relieved under some circumstances by decomposing states for the purpose of utility assignment.
- A nondeterministic action \( A \) may have several outcome states \( \text{Result}_i(A) \) indexed by the different outcomes of \( A \).
- Prior to executing an action \( A \), the agent assigns a probability \( P(\text{Result}_i(A) | \text{Do}(A), E) \) to each outcome (here \( E \) summarizes the agent’s evidence about the world).

2. The Basis of Utility Theory

- As a justification for the MEU principle, some constraints are imposed on the preferences that a rational agent may possess.
- In utility theory, different attainable outcomes (prizes) and the respective probabilities (chances) are formalized as lotteries:
  - A lottery \( L \) having outcomes \( A_1, \ldots, A_n \) with probabilities \( p_1 + \ldots + p_n = 1 \) is written as \( [p_1, A_1; \ldots; p_n, A_n] \).
  - A lottery \( [1, A] \) with a single outcome is abbreviated as \( A \).
- Preference relations for lotteries (or states) \( A \) and \( B \):
  - \( A \succ B \iff A \) is preferred to \( B \),
  - \( A \sim B \iff A \) is indifferent between \( A \) and \( B \), and
  - \( A \succeq B \iff A \succ B \) or \( A \sim B \).
Axioms of Utility Theory

For any lotteries $A$, $B$, and $C$:

1. **Orderability:** $(A \succ B) \lor (B \succ A) \lor (A \sim B)$
2. **Transitivity:** $(A \succ B) \land (B \succ C) \Rightarrow (A \succ C)$
3. **Continuity:** $A \succ B \succ C \Rightarrow \exists p[p, A; 1 - p, C] \sim B$
4. **Substitutability:** $A \sim B \Rightarrow [p, A; 1 - p, C] \sim [p, B; 1 - p, C]$
5. **Monotonicity:**
   
   $A \succ B \Rightarrow (p \geq q \Leftrightarrow \left[p, A; 1 - p, B\right] \succeq [q, A; 1 - q, B])$
6. **Decomposability** (the “no fun in gambling” rule):
   
   $[p, A; 1 - p, q, B; 1 - q, C] \sim [p, A; (1 - p)q, B; (1 - p)(1 - q), C]$

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The Utility of Money

- Utility theory has its roots in economy where the utility measure is **money** (an agent’s total net assets).
- Money plays a central role in human utility functions because of its almost universal exchangeability for all kinds of goods and services.
- Typically, there is a **monotonic preference** for money.
- Money behaves as a **value function** or **ordinal utility function**: more money is preferred to less when considering **definite amounts**.
- To understand monetary decision making under uncertainty we need to analyze the agent’s preferences between lotteries involving money.

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Beyond the axioms, an agent can have any preferences it likes.

**Example.** An agent prefers to have a prime number of euros in its bank account (having 16€ it would give away 3€).

- Preferences can also interact in complex ways.
  **Example.** Having a digital TV (in contrast to a conventional one) affects the preferences on soap operas one wishes to watch.

- We are interested in systematic ways of designing utility functions that generate the kinds of behavior we want.

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The existence of a utility function is guaranteed by the axioms:

1. **Utility principle:** if the axioms of utility theory are obeyed, then there is a real-valued function $U$ such that
   
   $U(A) > U(B) \iff A \succ B$ and
   
   $U(A) = U(B) \iff A \sim B$.
2. **Maximum Expected Utility principle:** the utility of a lottery
   
   $U([p_1, A_1; \ldots; p_n, A_n]) = \sum_i p_i U(A_i)$.

- However, the existence of a utility function $U$ need not imply the the agent is explicitly maximizing $U$ in its own deliberations.
- By observing an agent’s preferences, it is possible to construct a utility function representing what the agent is trying to achieve.
Example. A competitor in a TV game show is offered two prizes: either

A: 1000000€ for sure, or

B: after flipping a fair coin, either 3000000€ (heads) or 0€ (tails).

Is it irrational to choose the prize A?

1. The expected monetary values (EMV) of the choices are:
   
   \[ EMV(A) = 1 \times 1000000€ = 1000000€ \]
   
   \[ EMV(B) = 0.5 \times 3000000€ + 0.5 \times 0€ = 1500000€. \]

2. If \( S_k \) denotes the current wealth of \( k € \), expected utilities are:
   
   \[ EU(A) = U(S_{k+1000000}) \]
   
   \[ EU(B) = 0.5U(S_k) + 0.5U(S_{k+3000000}). \]

The choice depends on the respective utilities and \( k \) especially!

Example. St. Petersburg paradox [Bernoulli, 1738]: a fair coin is tossed \( n \) times until it comes up heads and the prize is \( 2^n € \).

How much would you pay for a chance to play this game?

- The expected monetary value for this game is
  \[ EMV = \sum P(Heads_i) \times 2^i = \sum_{i=1}^{\infty} \frac{1}{2} 2^i = \infty. \]
  
  A player should be willing to pay any finite sum!

- Bernoulli solved the paradox by setting \( U(S_{k+n}) = \log_2 n \):
  \[ EU = \sum P(Heads_i) \times U(Heads_i) = \sum_{i=1}^{\infty} \frac{1}{2} 2 = 2. \]

  A rational agent (with the given utility scale) should be willing to pay 4€ for playing the game, because \( U(S_{k+4}) = \log_2 4 = 2. \)

The utility of money is measured on a logarithmic scale (at least for positive amounts).

Grayson [1960] found an almost perfect fit to the logarithmic form.

- Mr. Beard’s preferences (a) turned out to be consistent with
  \[ U(S_{k+n}) = (22.09 \times \log(n + 150000) - 263.91) $. \]

Going into debt is usually considered disastrous.

- Preferences between different levels of debt (b) may be analogous (but reverse) to those of positive wealth.

Typically, for any lottery \( L \), the utility of being faced with \( L \) is less than the utility of being handed \( EMV(L) \) for sure.

- A risk-averse agent prefers a sure thing with a payoff that is less than the expected monetary value of a gamble.

- A desperately debted agent may behave in a risk-seeking way.

- A certainty equivalent of a lottery \( L \) is the sum that an agent is ready to accept as a substitute for participating in \( L \).

Example. The certainty equivalent is 400€ for a lottery \( L \) that gives 1000€ half the time and 0€ otherwise (\( EMV(L) = 500€ \)).

- An insurance is based on a positive insurance premium, i.e., the difference between \( EMV(L) \) and the certainty equivalent for \( L \).
Utility Scales and Assessment

- The axioms of utility do not specify a unique utility function.
  
  **Example.** For instance, two agents based on $U(S)$ and $U'(S) = k_1 + k_2 \times U(S)$ with $k_2 > 0$ behave identically.

- A way to assess utilities is to establish a scale with a "best possible prize" $u_{\text{max}}$ and a "worst possible catastrophe" $u_{\text{min}}$.

- Normalized utilities use a scale with $u_{\min} = 0$ and $u_{\max} = 1$.

- An intermediate utility $U(S) = p$ is determined by indifference between $S$ and a standard lottery $L = [p, U_{\text{max}}; (1 - p), U_{\text{min}}]$.

- Trade-offs in decision making let us assess the value of human life.

**Examples.** Micromort (1/100000 chance of death) and QALY (quality-adjusted life year) are measures for the value of human life.

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4. Multiattribute Utility Functions

- Multiattribute utility theory deals with utility functions $U(X_1, \ldots, X_n)$ that depend on several attributes $X_1, \ldots, X_n$.

- Each attribute $X_i$ ranges over discrete/continuous scalar values.

- For simplicity, it is assumed that (all other things being equal) greater values of an attribute $X_i$ correspond to higher utilities.

- We would like to identify regularities in the preference behavior as **representation theorems** for the corresponding utility functions:

$$U(x_1, \ldots, x_n) = f(f_1(x_1), \ldots, f_n(x_n))$$

where $f$ is a simple function such as addition.

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Dominance

- There is strict dominance of an option $S_1$ over other option $S_2$ if $S_1$ is better than $S_2$ with respect to all attributes.

  **Example.** An airport site $S_1$ costs less, generates less noise pollution, and is safer than another site $S_2$.

- Uncertain attribute values can be handled analogously.

- Strict dominance is useful in narrowing down the choices.

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Stochastic Dominance

**Example.** The costs of siting the airport at $S_1$ and $S_2$ are $3.7 \times 10^9\,€$ and $4.0 \times 10^9\,€$ with standard deviations $0.4 \times 10^9\,€$ and $0.35 \times 10^9\,€$.

- Knowing that the cost of $S_1$ is exactly $3.7 \times 10^9\,€$ does not enable decision making, because $S_2$ could be cheaper.

- But $S_1$ stochastically dominates $S_2 \Rightarrow S_2$ can be discarded.
Stochastic dominance is best detected from the respective cumulative probability distributions for the costs of $S_1$ and $S_2$:

If actions $A_1$ and $A_2$ lead to probability distributions $p_1(x)$ and $p_2(x)$ on attribute $X$, then $A_1$ stochastically dominates $A_2$ on $X$ if and only if for all $x$, $\int_{-\infty}^{x} p_1(y)dy \leq \int_{-\infty}^{x} p_2(y)dy$.

In many cases, stochastic dominance is easily detected. E.g., construction costs depend on the distance to the city center.

Preferences without Uncertainty

Attributes $X_1$ and $X_2$ are preferentially independent of a third attribute $X_3$ if the preference between outcomes $(x_1, x_2, x_3)$ and $(x'_1, x'_2, x_3)$ is independent of the particular value $x_3$ of $X_3$.

Mutual preferential independence (MPI) of $X_1, \ldots, X_n$: each pair of variables is preferentially independent from others.

If attributes $X_1, \ldots, X_n$ are mutually preferentially independent, then the agent’s behavior can be described as maximizing

$$V(S) = \sum_{i=1}^{n} V_i(X_i(S))$$

where each $V_i$ is a value function referring only to $X_i$.

A value function like $V(S)$ is called an additive value function.

Preferences with Uncertainty

Utility independence extends preferential independence to cover lotteries: a set of attributes $X$ is utility-independent of $Y$ if lotteries involving $X$ are independent of the particular values of $Y$.

A set of attributes $X$ is mutually utility-independent (MUI) if each subset $Y \subseteq X$ is utility-independent of $X-Y$.

If MUI holds, the agent’s behavior can be described in terms of a multiplicative utility function. For three attributes, $U_i = k_1 U_i + k_2 U_2 + k_3 U_3 + k_1 k_2 U_1 U_2 + k_2 k_3 U_2 U_3 + k_1 k_3 U_1 U_3 + k_1 k_2 k_3 U_1 U_2 U_3$ where $U_i$ denotes $U_i(X_i(S))$ for $i \in \{1, 2, 3\}$.

In general, an $n$-attribute problem exhibiting MUI can be modeled using $n$ single-attribute utilities and $n$ constants.

5. Decision Networks

Decision networks (or influence diagrams) extend Bayesian networks with additional nodes for actions and utilities:

1. Chance nodes (ovals) represent random variables with CPTs.
2. Decision nodes (rectangles) represent points where the decision-maker has a choice of actions to perform.
3. Utility nodes (diamonds) represent the agent’s utility function (a tabulation of the agent’s utility as a function of attributes).

Chance nodes (as well as utility nodes) may have both chance nodes and decision nodes as parents.

We concentrate on decision networks with a single decision node.
**Example.** Consider the airport siting problem. In addition to the choice being made, factors including **AirTraffic**, **Litigation**, and **Construction** affect utility indirectly via **Deaths**, **Noise**, and **Cost**.

![AirTraffic, Litigation, Construction network diagram]

The algorithm for evaluating a decision network in the following:

1. Set the evidence variables for the current state.
2. For each possible value of the decision node:
   a. Set the decision node to that value (like any evidence variable).
   b. Calculate the posterior probabilities for the parent nodes of the utility node using standard probabilistic inference algorithms.
   c. Calculate the resulting utility for the action.
3. Return the action with the highest utility.

➤ We will later consider the possibility of executing several actions in sequence which makes the problem much more interesting.

**Example.** The decision network for the airport siting problem can be simplified by factoring out chance nodes describing outcome states:

![Simplified network diagram]

➤ A way to simplify a decision network is to represent the expected utility of actions using **action-utility tables**.

Less flexible to update!

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**6. THE VALUE OF INFORMATION**

➤ One of the most important parts of decision making is knowing what questions to ask to obtain all relevant information.

**Example.** A doctor cannot expect to be provided with the results of all possible diagnostic tests when meeting a patient.

➤ The **value of information** is the difference between the expected utilities of the best actions before and after obtaining information.

➤ The acquisition of information is achieved by **sensing actions**.

➤ **Information value theory** is a form of sequential decision making.
Example. An oil company is willing to buy one of \( n \) indistinguishable blocks of ocean drilling rights. The setting is as follows:

1. There are \( n \) blocks for sale.
2. Exactly one block contains oil worth \( C \epsilon \).
3. The price of a single block is \( \frac{C}{n} \epsilon \).

A seismologist offers the company the results of a survey of block 3.

- How much is the company willing to pay for knowing the results?

- The expected value of this piece of information is
  \[
  \frac{1}{n} (C - \frac{C}{n}) + \frac{n-1}{n} \left( \frac{C}{n-1} - \frac{C}{n} \right) = \frac{C}{n} \quad (\epsilon).
  \]

- The information is is worth as much as the block itself!

A General Formula

- It is expected that the exact value of some random variable \( E_j \) is obtained: hence the term value of perfect information (VPI).
- The utility \( EU(\alpha | E) \) of the current best action \( \alpha \) is defined by
  \[
  \max_A \sum_t U(\text{Result}_t(A))P(\text{Result}_t(A) \mid E, Do(A)).
  \]
- Given a piece of evidence \( E_j \) this becomes
  \[
  EU(\alpha_{E_j} \mid E, E_j) = \max_A \sum_t U(\text{Result}_t(A))P(\text{Result}_t(A) \mid E, Do(A), E_j).
  \]
- But the value of \( E_j \) is currently unknown, and we have to average over all possible values \( e_{jk} \) of \( E_j \). Thus
  \[
  VPI(E_j) = \left( \sum_k P(E_j = e_{jk} \mid E)EU(\alpha_{e_{jk}} \mid E, E_j = e_{jk}) \right) - EU(\alpha \mid E).
  \]

Properties of the Value of Information

- "Information has value to the extent that it is likely to cause a change of plan, and to the extent that the new plan will be significantly better than the old plan".

Example. Consider different routes through a mountain range.

(a) A straight highway through a low pass (action \( A_1 \)) is clearly preferable to a winding dirt road over the top (action \( A_2 \)).

(b) The choice between two different winding dirt roads of slightly different lengths – each of which may be blocked or not.

(c) The differences are likely to be small in summertime.

Additional information becomes valuable in the case (b).
Implementing an Information-Gathering Agent

- For now, it is assumed that with each observable evidence variable $E_j$, there is an associated cost $\text{Cost}(E_j)$ of obtaining $E_j$ via tests.
- An information gathering agent should request the most valuable piece of information $E_j$ compared to $\text{Cost}(E_j)$:

```plaintext
function INFORMATION-GATHERING-AGENT(perceptron) returns an action
    static D: a decision network
    integrate perceptron into D
    if $\text{VPI}(E_j)$ > $\text{Cost}(E_j)$
        then return REQUEST(E_j)
    else return the best action from D
```

- The procedure implements myopic information gathering, since \( \text{VPI} \) is short-sightedly applied to single pieces of evidence.

Summary

- Decision theory = probability theory + utility theory.
- A rational agent considers all possible actions and chooses the one that leads to the best expected outcome.
- Decision networks – a generalization of Bayesian networks – provide a simple formalism for expressing and solving decision problems.
- The value of information is defined as the expected improvement in utility compared to making a decision without the information.
- Expert systems that incorporate utility information have additional capabilities compared to pure inference systems.

Decision-Theoretic Expert Systems

The knowledge engineering process for a decision-theoretic system:

1. Determine the scope of the problem (decide nodes).
2. Lay out the topology of the network (analyze dependencies).
3. Assign probabilities to chance nodes.
4. Assign utilities to utility nodes.
5. Enter available evidence to the network.
6. Evaluate posterior probabilities and utilities for the nodes.
7. Gather new evidence using value of information as a criterion.
8. Perform sensitivity analysis for the assigned probabilities/utilities.

Questions

Recall the domain of soccer playing agents and formalize a ball tracking system using a Bayesian network with the following variables:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Values</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tired</td>
<td>True, False</td>
<td>Is the agent feeling tired?</td>
</tr>
<tr>
<td>Angle</td>
<td>Left, Center, Right</td>
<td>Angle with respect to the ball</td>
</tr>
<tr>
<td>Distance</td>
<td>Far, Close, Touch</td>
<td>Distance to the ball</td>
</tr>
</tbody>
</table>

- For each variable $X$ of these, introduce an additional variable $X_{\text{next}}$ referring to the outcome of actions available to the agent:

  TurnLeft, TurnRight, Run, and Nop.

- Add a utility node that depends on $Tired_{\text{next}}$, $Angle_{\text{next}}$, and $Distance_{\text{next}}$. Define a utility function based on these attributes.