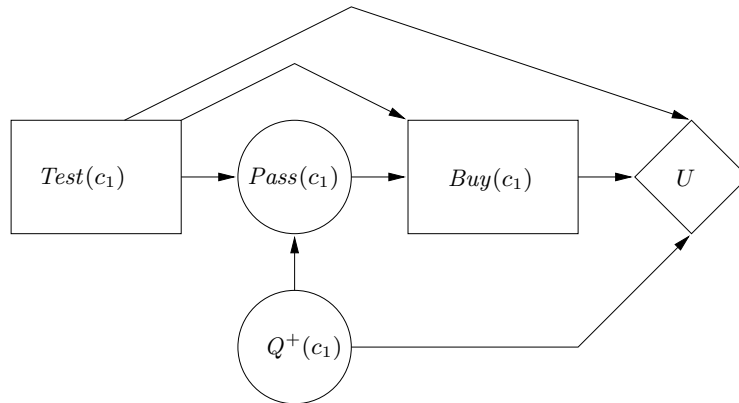


1. The following Boolean variables are introduced to model the domain

- $Q^+(c_1)$ : The car  $c_1$  is in good shape.
- $Pass(c_1)$ : The car  $c_1$  passed the test  $t_1$ .
- $Test(c_1)$ : The buyer orders the test  $t_1$  on  $c_1$ .
- $Buy(c_1)$ : The buyer decides to buy  $c_1$ .

The last two are interpreted as pairs of actions:  $\neg test(c_1)$  means that the buyer chooses to skip the test  $t_1$  on  $c_1$  whereas  $\neg buy(c_1)$  means that the buyer decides not to buy  $c_1$ .

(a) The following decision network models the used-car buyer's dilemma:



However, this is out of scope in view of decision networks involving only one decision node. One possibility is to reduce the network given in the figure according to the two values of  $Test(c_1)$ . If it is false, then drop the nodes for  $Test(c_1)$  and  $Pass(c_1)$ , and all arrows associated with them. Otherwise, drop  $Test(c_1)$  and the respective arrows.

(b) Given  $\neg test(c_1)$ , the expected utilities for the buyer are

$$\begin{aligned}
 EU(buy(c_1)) &= P(q^+(c_1))U(q^+(c_1)) + P(\neg q^+(c_1))U(\neg q^+(c_1)) \\
 &= 0.7 \times (2000\text{€} - 1500\text{€}) + \\
 &\quad 0.3 \times (2000\text{€} - 1500\text{€} - 700\text{€}) \\
 &= 350\text{€} - 60\text{€} = 290\text{€}.
 \end{aligned}$$

and  $EU(\neg buy(c_1)) = 0\text{€}$ .

(c) Using Bayes' theorem, we obtain

$$\begin{aligned}
& P(q^+(c_1) \mid \text{pass}(c_1)) \\
&= \beta P(\text{pass}(c_1) \mid q^+(c_1))P(q^+(c_1)) \\
&= \beta \times 0.80 \times 0.70 = \beta \times 0.56 = \frac{0.56}{0.665} \approx 0.8421
\end{aligned}$$

$$\begin{aligned}
& P(\neg q^+(c_1) \mid \text{pass}(c_1)) \\
&= \beta P(\text{pass}(c_1) \mid \neg q^+(c_1))P(\neg q^+(c_1)) \\
&= \beta \times 0.35 \times 0.30 = \beta \times 0.105 = \frac{0.105}{0.665} \approx 0.1579
\end{aligned}$$

$$\begin{aligned}
& P(q^+(c_1) \mid \neg \text{pass}(c_1)) \\
&= \gamma P(\neg \text{pass}(c_1) \mid q^+(c_1))P(q^+(c_1)) \\
&= \gamma \times 0.20 \times 0.70 = \gamma \times 0.14 = \frac{0.14}{0.335} \approx 0.4179
\end{aligned}$$

$$\begin{aligned}
& P(\neg q^+(c_1) \mid \neg \text{pass}(c_1)) \\
&= \gamma P(\neg \text{pass}(c_1) \mid \neg q^+(c_1))P(\neg q^+(c_1)) \\
&= \gamma \times 0.65 \times 0.30 = \gamma \times 0.195 = \frac{0.195}{0.335} \approx 0.5821
\end{aligned}$$

where  $\beta = \frac{1}{0.665}$  and  $\gamma = \frac{1}{0.335}$  are normalizing constants so that  $P(\text{pass}(c_1)) = \frac{1}{\beta} = 0.665$  and  $P(\neg \text{pass}(c_1)) = \frac{1}{\gamma} = 0.335$ .

(d) The expected utilities are

$$\begin{aligned}
& \text{EU}(\text{buy}(c_1) \mid \text{pass}(c_1)) \\
&= P(q^+(c_1) \mid \text{pass}(c_1))U(q^+(c_1)) + \\
&\quad P(\neg q^+(c_1) \mid \text{pass}(c_1))U(\neg q^+(c_1)) \\
&= 0.8421 \times 500\text{€} + 0.1579 \times (-200\text{€}) \\
&= 421.05\text{€} - 31.58\text{€} = 389.47\text{€}.
\end{aligned}$$

$$\begin{aligned}
& \text{EU}(\text{buy}(c_1) \mid \neg \text{pass}(c_1)) \\
&= P(q^+(c_1) \mid \neg \text{pass}(c_1))U(q^+(c_1)) + \\
&\quad P(\neg q^+(c_1) \mid \neg \text{pass}(c_1))U(\neg q^+(c_1)) \\
&= 0.4179 \times 500\text{€} + 0.5821 \times (-200\text{€}) \\
&= 208.95\text{€} - 116.42\text{€} = 92.53\text{€}
\end{aligned}$$

whereas for not buying they are 0€.

(e) Since  $\text{buy}(c_1)$  is the best action in any case, we obtain

$$\begin{aligned}
& \text{VPI}(\text{Pass}(c_1)) \\
&= P(\text{pass}(c_1))\text{EU}(\text{buy}(c_1) \mid \text{pass}(c_1)) + \\
&\quad P(\neg \text{pass}(c_1))\text{EU}(\text{buy}(c_1) \mid \neg \text{pass}(c_1)) - \\
&\quad \text{EU}(\text{buy}(c_1)) \\
&= 0.665 \times 389.47\text{€} + 0.335 \times 92.53\text{€} - 290\text{€} \approx 0\text{€}.
\end{aligned}$$

Thus performing the test  $t_1$  on  $c_1$  does not bring any additional value for the buyer and in particular, given the price of the test (50€).

2. The goal is to prove in general that

$$\text{VPI}_E(X) = \left( \sum_x P(x \mid E)\text{EU}(\alpha_x \mid E, x) \right) - \text{EU}(\alpha \mid E) \geq 0 \quad (1)$$

where  $\alpha$  is the best current action given  $E$ . Its utility is defined by

$$\text{EU}(\alpha | E) = \max_A \text{EU}(A | E) = \max_A \sum_i U(R_i(A))P(R_i(A) | E)$$

and each  $\alpha_x$  is similarly defined for evidence consisting of  $E$  and  $x$ .

$$\begin{aligned} \text{EU}(\alpha | E) &= \max_A \sum_i U(R_i(A))P(R_i(A) | E) \\ &= \max_A \sum_i U(R_i(A)) \sum_x P(R_i(A) | E, x)P(x | E) \\ &= \max_A \sum_i \sum_x P(x | E)U(R_i(A))P(R_i(A) | E, x) \\ &= \max_A \sum_x \sum_i P(x | E)U(R_i(A))P(R_i(A) | E, x) \\ &= \max_A \sum_x P(x | E) \sum_i U(R_i(A))P(R_i(A) | E, x) \\ &\leq \sum_x \max_A P(x | E) \sum_i U(R_i(A))P(R_i(A) | E, x) \\ &= \sum_x P(x | E) \max_A \sum_i U(R_i(A))P(R_i(A) | E, x) \\ &= \sum_x P(x | E)\text{EU}(\alpha_x | E, x). \end{aligned}$$

Now (1) follows by this inequality so that  $\text{VPI}_E(X) \geq 0$ . Note that  $E$  can be replaced by a vector  $\mathbf{E}$  if appropriate and hence  $\text{Do}(A)$  can be viewed as a part of  $\mathbf{E}$  (c.f. R&N).