T-79.5101 Spring 2008
Advanced Course in Computational Logic
Exercise Session 8
Solutions

1. **KB** is the set of symmetric frames. Let's translate the formula in the exercise into predicate logic:

$$\begin{aligned}
&\tau(\neg\Box\neg\Box\neg\Box\neg P \rightarrow \neg\Box\neg P, x)\\
&= \tau(\neg\Box\neg\Box\neg\Box\neg P, x) \rightarrow \tau(\neg\Box\neg P, x)\\
&= \neg\tau(\Box\neg\Box\neg\Box\neg P, x) \rightarrow \neg\tau(\Box\neg P, x)\\
&= \neg\forall y\big(R(x,y) \rightarrow \tau(\neg\Box\neg\Box\neg P, y)\big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \tau(\neg P, y)\big)\\
&= \neg\forall y\big(R(x,y) \rightarrow \neg\tau(\Box\neg\Box\neg P, y)\big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg\tau(P, y)\big)\\
&= \neg\forall y\Big(R(x,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \tau(\neg\Box\neg P, x)\big)\Big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\\
&= \neg\forall y\Big(R(x,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\tau(\Box\neg P, x)\big)\Big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\\
&= \neg\forall y\Big(R(x,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \tau(\neg P, y)\big)\big)\Big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\\
&= \neg\forall y\Big(R(x,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg\tau(P, y)\big)\big)\Big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\\
&= \neg\forall y\Big(R(x,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big)\Big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\\
&= \varphi
\end{aligned}$$

Additionally, we encode the frame axiom in predicate logic:

$$\forall x\forall y\big(R(x,y) \rightarrow R(y,x)\big) \qquad \text{(symmetricity)}$$

Then, we construct a complete tableau starting from the frame axiom and $\neg\forall x\varphi$.

$$\begin{array}{lll}
1. & \forall x\forall y\big(R(x,y) \rightarrow R(y,x)\big) & \\
2. & \neg\forall x\Big(\neg\forall y\Big(R(x,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big)\Big) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\Big) & \\
3. & \neg\Big(\neg\forall y\Big(R(c,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big)\Big) \rightarrow \neg\forall y\big(R(c,y) \rightarrow \neg P(y)\big)\Big) & (2,\ x/c)\\
4. & \neg\forall y\Big(R(c,y) \rightarrow \neg\forall x\big(R(y,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big)\Big) & (3)\\
5. & \neg\neg\forall y\big(R(c,y) \rightarrow \neg P(y)\big) & (3)\\
6. & \neg\Big(R(c,d) \rightarrow \neg\forall x\big(R(d,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big)\Big) & (4,\ y/d)\\
7. & \forall y\big(R(c,y) \rightarrow \neg P(y)\big) & (5)\\
8. & R(c,d) & (6)\\
9. & \neg\neg\forall x\big(R(d,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big) & (6)\\
10. & \forall x\big(R(d,x) \rightarrow \neg\forall y\big(R(x,y) \rightarrow \neg P(y)\big)\big) & (9)\\
11. & R(d,c) \rightarrow \neg\forall y\big(R(c,y) \rightarrow \neg P(y)\big) & (10,\ x/c)
\end{array}$$

$$\begin{array}{l|l}
\begin{array}{lll}
12. & \neg R(d,c) & (11)\\
14. & \forall y\big(R(c,y) \rightarrow R(y,c)\big) & (1,\ x/c)\\
15. & R(c,d) \rightarrow R(d,c) & (14,\ y/d)
\end{array}
&
\begin{array}{lll}
13. & \neg\forall y\big(R(c,y) \rightarrow \neg P(y)\big) & (11)\\
18. & \neg\big(R(c,e) \rightarrow \neg P(e)\big) & (13,\ y/e)\\
19. & R(c,e) & (18)\\
20. & \neg\neg P(e) & (18)\\
21. & R(c,e) \rightarrow \neg P(e) & (7,\ y/e)
\end{array}
\\
\begin{array}{l|l}
16. \neg R(c,d)\ (15) & 17.\ R(d,c)\ (15)\\
\quad\otimes & \quad\otimes
\end{array}
&
\begin{array}{l|l}
22. \neg R(c,e)\ (21) & 23. \neg P(e)\ (21)\\
\quad\otimes & \quad\otimes
\end{array}
\end{array}$$

2. a) $\mathcal{M}, s_1 \Vdash K_1P$ since $v(s_1, P) = v(s_2, P) =$ true, and $\mathcal{M}, s_1 \Vdash K_2P$ and $\mathcal{M}, s_1 \Vdash K_3P$ since $v(s_1, P) =$ true. Thus $\mathcal{M}, s_1 \Vdash EP$.

   b) $\mathcal{M}, s_1 \Vdash K_2EP$ and $\mathcal{M}, s_1 \Vdash K_3EP$ since $\mathcal{M}, s_1 \Vdash EP$. Furthermore, $\mathcal{M}, s_2 \Vdash K_1P$ since $v(s_1, P) = v(s_2, P) =$ true, $\mathcal{M}, s_2 \Vdash K_2P$ since $v(s_2, P) = v(s_3, P) =$ true, and $\mathcal{M}, s_2 \Vdash K_3P$ since $v(s_2, P) =$ true. Hence $\mathcal{M}, s_2 \Vdash EP$, and it follows that $\mathcal{M}, s_1 \Vdash K_1EP$. Thus $\mathcal{M}, s_1 \Vdash EEP$.

   This result can also be obtained by noticing that P is true in all worlds that are reachable from s_1 in two steps.

   c) $\mathcal{M}, s_1 \nVdash CP$ since s_4 is C-reachable from s_1 and $v(s_4, P) =$ false.

3. Recall that a model is universal of it is based on a universal frame. **S5** denotes the set of all universal frames. Let φ be a formula that is not **S5**-valid, where Then there is a universal countermodel for $\mathcal{M} = \langle S, R, v\rangle$ having a world $s \in S$ for which $\mathcal{M}, s \nVdash \varphi$.

   Let $F = \{\Box\psi \mid \Box\psi$ is a subformula of φ and $\mathcal{M}, s \Vdash \neg\Box\psi\}$.

   Then for each formula $\Box\psi \in F$ there is a corresponding $s_\psi \in S$ such that $\langle s, s_\psi\rangle \in R$ and $\mathcal{M}, s_\psi \Vdash \neg\psi$.

   Let $\mathcal{M}' = \langle S', R', v'\rangle$, where $S' = \{s\} \cup \{s_\psi \mid \Box\psi \in F\} \subseteq S$, $R' = S' \times S'$ and $v'(s', P) = v(s', P)$ for all $s' \in S'$ and for all atomic formulas P occurring in φ.

   We will show by induction that for every $s' \in S'$ and for every subformula ψ of φ it holds that $\mathcal{M}, s' \Vdash \psi \quad \text{iff} \quad \mathcal{M}', s' \Vdash \psi$.

   The base case (for every atomic formula) is trivial. Furthermore, the induction step follows immediately for all subformulas of the form $\psi' \wedge \psi''$ and $\neg\psi$. Now consider a subformula of the form $\Box\psi$. Let $s' \in S'$.

   If $\mathcal{M}, s' \Vdash \Box\psi$, then $\mathcal{M}, t \Vdash \psi$ for all $t \in S'$ due to the fact that $\mathcal{M}$ is universal. By the induction hypothesis, $\mathcal{M}', t \Vdash \psi$ for all $t \in S'$. It follows that $\mathcal{M}', s' \Vdash \Box\psi$.

   On the other hand, if $\mathcal{M}, s' \nVdash \Box\psi$, then $\mathcal{M}, t \Vdash \neg\Box\psi$ for all $t \in S'$ since $\mathcal{M}$ is universal. Especially $\mathcal{M}, s \Vdash \neg\Box\psi$, and hence $\Box\psi \in F$. Now we know that there is a world $s_\psi \in S'$ such that $\langle s, s_\psi\rangle \in R$ and $\mathcal{M}, s_\psi \Vdash \neg\psi$, that is, $\mathcal{M}, s_\psi \nVdash \psi$. By the induction hypothesis it follows that $\mathcal{M}', s_\psi \nVdash \psi$, and thus $\mathcal{M}', s_\psi \Vdash \neg\psi$. Since $\mathcal{M}'$ is universal, we have $\mathcal{M}', s' \nVdash \Box\psi$. Especially $s \in S'$ and $\mathcal{M}, s \nVdash \varphi$, and thus by the above it follows that $\mathcal{M}', s \nVdash \varphi$. Hence $\mathcal{M}'$ is a countermodel for φ, as well.

If φ is not **S5**-valid, there is a universal countermodel $\mathcal{M}$ and a world s such that $\mathcal{M}, s \nVdash \varphi$. By our construction we obtain another universal countermodel $\mathcal{M}'$ for φ having at most $|\mathrm{Sub}(\varphi)|$ worlds.

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Exercise Session 9
Solutions

1. Basic operator:

$$M, s \models \mathbf{AX}P \text{ iff } \mathcal{M}, t \models P \text{ for all } t \text{ such that } sRt$$

Replace P by $\neg P$:

$$\begin{aligned}
&\mathcal{M}, s \models \mathbf{AX}\neg P \text{ iff } \mathcal{M}, t \models \neg P \text{ for all } t \text{ such that } sRt\\
&\mathcal{M}, s \not\models \mathbf{AX}\neg P \text{ iff } \mathcal{M}, t \not\models \neg P \text{ for some } t \text{ such that } sRt\\
&\mathcal{M}, s \models \neg\mathbf{AX}\neg P \text{ iff } \mathcal{M}, t \not\models \neg P \text{ for some } t \text{ such that } sRt\\
&\mathcal{M}, s \models \mathbf{EX}P \text{ iff } \mathcal{M}, t \models P \text{ for some } t \text{ such that } sRt
\end{aligned}$$

Basic operator:

$$\begin{aligned}
\mathcal{M}, s \models \mathbf{A}(P\mathbf{U}Q) \text{ iff } & \text{for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s\\
& \text{in } \mathcal{M}, \text{ there is some } i \text{ such that } \mathcal{M}, s_i \models Q,\\
& \text{and for all } j < i \text{ it holds that } \mathcal{M}, s_j \models P.
\end{aligned}$$

Make the substitutions $P \to \top$, $Q \to P$:

$$\begin{aligned}
\mathcal{M}, s \models \mathbf{A}(\top\mathbf{U}P) \text{ iff } & \text{for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s\\
& \text{in } \mathcal{M} \text{ there is some } i \text{ such that } \mathcal{M}, s_i \models P\\
& \text{and for all } j < i \text{ it holds that } \mathcal{M}, s_j \models \top.\\
\mathcal{M}, s \models \mathbf{AF}P \text{ iff } & \text{for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s\\
& \text{in } \mathcal{M} \text{ there is some } i \text{ such that } \mathcal{M}, s_i \models P.
\end{aligned}$$

Basic operator:

$$\begin{aligned}
\mathcal{M}, s \models \mathbf{E}(P\mathbf{U}Q) \text{ iff } & \text{there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s\\
& \text{in } \mathcal{M} \text{ and there is some } i \text{ such that } \mathcal{M}, s_i \models Q\\
& \text{and for all } j < i \text{ it holds that } \mathcal{M}, s_j \models P.
\end{aligned}$$

Make the substitutions $P \to \top$, $Q \to P$:

$$\begin{aligned}
&\mathcal{M}, s \models \mathbf{E}(\top \mathbf{U} P) \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and there is some } i \text{ such that } \mathcal{M}, s_i \models P \\
&\qquad \text{and for all } j < i \text{ it holds that } \mathcal{M}, s_j \models \top. \\
&\mathcal{M}, s \models \mathbf{EF} P \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and there is some } i \text{ such that } \mathcal{M}, s_i \models P.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}, s \models \mathbf{EF} P \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and there is some } i \text{ such that } \mathcal{M}, s_i \models P. \\
&\mathcal{M}, s \models \mathbf{EF} \neg P \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and there is some } i \text{ such that } \mathcal{M}, s_i \models \neg P. \\
&\mathcal{M}, s \not\models \mathbf{EF} \neg P \text{ iff for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and for all } i \text{ it holds that } \mathcal{M}, s_i \not\models \neg P. \\
&\mathcal{M}, s \models \neg \mathbf{EF} \neg P \text{ iff for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and for all } i \text{ it holds that } \mathcal{M}, s_i \not\models \neg P. \\
&\mathcal{M}, s \models \mathbf{AG} P \text{ iff for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ and for all } i \text{ it holds that } \mathcal{M}, s_i \models P.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}, s \models \mathbf{AF} P \text{ iff for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ there is some } i \text{ such that } \mathcal{M}, s_i \models P. \\
&\mathcal{M}, s \models \mathbf{AF} \neg P \text{ iff for all full paths } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ there is some } i \text{ such that } \mathcal{M}, s_i \models \neg P. \\
&\mathcal{M}, s \not\models \mathbf{AF} \neg P \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ such that for all } i \text{ it holds that } \mathcal{M}, s_i \not\models \neg P. \\
&\mathcal{M}, s \models \neg \mathbf{AF} \neg P \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ such that for all } i \text{ it holds that } \mathcal{M}, s_i \not\models \neg P. \\
&\mathcal{M}, s \models \mathbf{EG} P \text{ iff there is a full path } (s_0, s_1, \dots) \text{ with } s_0 = s \\
&\qquad \text{in } \mathcal{M} \text{ such that for all } i \text{ it holds that } \mathcal{M}, s_i \models P.
\end{aligned}$$

2.

$$\begin{aligned}
&\mathcal{M}, x \models P \mathbf{U} Q \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models Q \\
&\qquad \text{and for all } j < i \text{ it holds that } \mathcal{M}, x^j \models P. \\
&\mathcal{M}, x \models \top \mathbf{U} P \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models P \\
&\qquad \text{and for all } j < i \text{ it holds that } \mathcal{M}, x^j \models \top \\
&\mathcal{M}, x \models \mathbf{F} P \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models P.
\end{aligned}$$

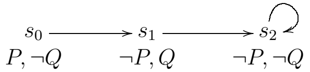
$$\begin{aligned}
&\mathcal{M}, x \models \mathbf{F} P \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models P. \\
&\mathcal{M}, x \models \mathbf{F} \neg P \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models \neg P. \\
&\mathcal{M}, x \not\models \mathbf{F} \neg P \text{ iff for all } i \text{ it holds that } \mathcal{M}, x^i \not\models \neg P. \\
&\mathcal{M}, x \models \neg \mathbf{F} \neg P \text{ iff for all } i \text{ it holds that } \mathcal{M}, x^i \not\models \neg P. \\
&\mathcal{M}, x \models \mathbf{G} P \text{ iff for all } i \text{ it holds that } \mathcal{M}, x^i \models P.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{M}, x \models P \mathbf{U} Q \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models Q \\
&\qquad \text{and for all } j < i \text{ it holds that } \mathcal{M}, x^j \models P. \\
&\mathcal{M}, x \models (\neg P) \mathbf{U} (\neg Q) \text{ iff there is some } i \text{ such that } \mathcal{M}, x^i \models \neg Q \\
&\qquad \text{and for all } j < i \text{ it holds that } \mathcal{M}, x^j \models \neg P. \\
&\mathcal{M}, x \not\models (\neg P) \mathbf{U} (\neg Q) \text{ iff for all } i: \\
&\qquad \mathcal{M}, x^i \not\models \neg Q \text{ or there is some } j < i \text{ such that } \mathcal{M}, x^j \not\models \neg P. \\
&\mathcal{M}, x \models \neg\big((\neg P) \mathbf{U} (\neg Q)\big) \text{ iff for all } i: \\
&\qquad \text{if } \mathcal{M}, x^i \models \neg Q, \text{ then there is some } j < i \text{ such that } \mathcal{M}, x^j \not\models \neg P. \\
&\mathcal{M}, x \models P \mathbf{R} Q \text{ iff for all } i: \\
&\qquad \text{if } \mathcal{M}, x^i \not\models Q, \text{ then there is some } j < i \text{ such that } \mathcal{M}, x^j \models P.
\end{aligned}$$

3. For example, define

$$\begin{array}{ll}
v(s_0, P) = \text{true} & v(s_0, Q) = \text{false} \\
v(s_1, P) = \text{false} & v(s_1, Q) = \text{true} \\
v(s_2, P) = \text{false} & v(s_2, Q) = \text{false}.
\end{array}$$

Then we have the model



Now for the full path $x = (s_0, s_1, s_2, s_2, s_2, \ldots)$ it holds that

$$\mathcal{M}, x \models P\mathbf{U}Q, \text{ since } \mathcal{M}, x^1 \models Q \text{ and } \mathcal{M}, x^j \models P \text{ holds for all } j < 1,$$

but $\mathcal{M}, x \not\models Q\mathbf{R}P$ since $\mathcal{M}, x^1 \not\models P$ and there is no $j < 1$ for which $\mathcal{M}, x^j \models Q$.

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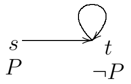
Advanced Course in Computational Logic
Exercise Session 10
Solutions

1. a) $P \wedge \mathbf{EF}Q$
   b) $\mathbf{EF}(P \wedge \mathbf{AXAG}\neg P)$
   c) $\mathbf{AG}\big(P \rightarrow \mathbf{AX}(P \rightarrow \mathbf{EF}Q)\big)$
   d) $\big(P \rightarrow \mathbf{A}(P\mathbf{U}Q)\big) \wedge \big(\neg P \rightarrow \mathbf{AX}(P \vee \mathbf{AX}P)\big)$
   e) $\mathbf{E}\big(P\mathbf{U}\mathbf{AG}((Q \rightarrow \mathbf{AX}\neg Q) \wedge (\neg Q \rightarrow \mathbf{AX}Q))\big)$
   f) $\mathbf{AG}\big(P \rightarrow \mathbf{AG}(\neg Q \wedge \neg R)\big) \wedge \mathbf{AG}\big((Q \vee R) \rightarrow \mathbf{AG}\neg P\big)$

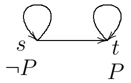
2. a) $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \big\{\langle s, t\rangle, \langle t, t\rangle\big\}$, $v(s, P) =$ true and $v(t, P) =$ false.



$\mathcal{M}, s \models \mathbf{AF}P$ holds since $(s, t, t, t, \ldots)$ is the only full path beginning from s and on this path there is a state s such that $\mathcal{M}, s \models P$ holds. Thus $\mathbf{AF}P$ is satisfiable.

For $\mathbf{GF}P$ to be satisfiable in the model $\mathcal{M}$ there should be a full path x in $\mathcal{M}$ for which $\mathcal{M}, x \models \mathbf{GF}P$. Then $\mathcal{M}, x^i \models \mathbf{F}P$ should hold for all $i \geq 0$, that is, for all $i \geq 0$ there should be a $j \geq i$ such that $\mathcal{M}, x^j \models P$. In other words, P should be true on infinitely many (infinite) suffixes of the path x However, there are no such paths since the only full paths in $\mathcal{M}$ are $(s, t, t, t, \ldots)$ and $(t, t, t, \ldots)$, and P is true only on finitely many suffixes of these paths. Thus the formula $\mathbf{GF}P$ is not satisfiable in the model $\mathcal{M}$.

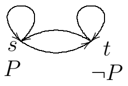
   b) $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \big\{\langle s, s\rangle, \langle s, t\rangle, \langle t, t\rangle\big\}$, $v(s, P) =$ false and $v(t, P) =$ true.



$\mathcal{M}, s \models \mathbf{EFAG}P$ and $\mathcal{M}, t \models \mathbf{EFAG}P$ hold since the model includes the full paths $(s, t, t, t, \ldots)$ and $(t, t, t, \ldots)$ which go though the state t, and clearly $\mathcal{M}, t \models \mathbf{AG}P$. Thus the formula $\mathbf{EFAG}P$ is valid in the model.

However, $\mathbf{FG}P$ is not valid in the model: the full path $(s, s, s, \ldots)$ has no infinite suffix x such that $\mathcal{M}, x^i \models P$ holds for all i (since $v(s, P) = \text{false}$), and hence $\mathcal{M}, (s, s, s, \ldots) \not\models \mathbf{FG}P$.

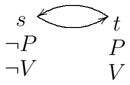
c) $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \big\{\langle s, s\rangle, \langle s, t\rangle, \langle t, s\rangle, \langle t, t\rangle\big\}$, $v(s, P) = \text{true}$ and $v(t, P) = \text{false}$.



The formula $\mathbf{FX}P$ is satisfiable since the model has (for example) the full path $x = (t, s, s, s, \ldots)$ for which $\mathcal{M}, x \models \mathbf{X}P$ (since $v(s, P) = \text{true}$), and hence $\mathcal{M}, x \models \mathbf{FX}P$.

However, the formula $\mathbf{EFAX}P$ is not satisfiable in any state of the model: otherwise, there should be a full path that begins from s or t which goes through states s and t which would also go though a state which satisfies $\mathbf{AX}P$. In other words, either $\mathcal{M}, s \models \mathbf{AX}P$ or $\mathcal{M}, t \models \mathbf{AX}P$ should holds; however, this is not the case since both s and t have a successor (t) in R for which $\mathcal{M}, t \not\models P$.

3. a) $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \big\{\langle s, t\rangle, \langle t, s\rangle\big\}$, $v(s, P) = v(s, V) = \text{false}$ and $v(t, P) = v(t, V) = \text{true}$.

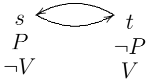


Here we can separately look at the paths $x_1 = (s, t, s, t, \ldots)$ and $x_2 = (t, s, t, s, \ldots)$.

- $\mathcal{M}, s \models \mathbf{E}(\neg V \mathbf{U} P)$ holds since $\mathcal{M}, x_1^1 \models P$ (because $v(t, P) = \text{true}$), and for all $i < 1$ we have $\mathcal{M}, x_1^i \models \neg V$. Furthermore, $\mathcal{M}, t \models \mathbf{E}(\neg V \mathbf{U} P)$ holds since the full path x_2 starts from t and $\mathcal{M}, x_2^0 \models P$.
- Since $\mathcal{M}, x_1^0 \models \neg P$, we have $\mathcal{M}, s \models \mathbf{E}(V \mathbf{U} \neg P)$. Similarly, $\mathcal{M}, t \models \mathbf{E}(V \mathbf{U} \neg P)$ holds since $\mathcal{M}, x_2^1 \models \neg P$ and $\mathcal{M}, x_2^i \models V$ for all $i < 1$.
- $\mathcal{M}, s \models \mathbf{AF}(V \rightarrow \mathbf{AX}\neg V) \wedge \mathbf{EF}V$ since x_1 is the only path that starts from s and $\mathcal{M}, x_1 \models \mathbf{F}(V \rightarrow \mathbf{AX}\neg V)$ (because, e.g., $\mathcal{M}, x_1^0 \models V \rightarrow \mathbf{AX}\neg V$ since $v(s, V) = \text{false}$) and, additionally, $\mathcal{M}, x_1 \models \mathbf{F}V$ since the path x_1 goes through the state t and $v(t, V) = \text{true}$.

Similarly, we have $\mathcal{M}, t \models \mathbf{AF}(V \rightarrow \mathbf{AX}\neg V) \wedge \mathbf{EF}V$ since x_2 is the only full path that starts from t and $\mathcal{M}, x_2^0 \models \mathbf{AX}\neg V$ holds since $\mathcal{M}, t \models \mathbf{AX}\neg V$ (the only successor of t is s and $v(s, V) = \text{false}$). Furthermore, $\mathcal{M}, t \models \mathbf{EF}V$ holds since for the full path x_2 that starts from t we have $\mathcal{M}, x_2^0 \models V$ (because $v(t, V) = \text{true}$).

b) $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \big\{\langle s, t\rangle, \langle t, s\rangle\big\}$, $v(t, P) = v(s, V) = \text{false}$ and $v(s, P) = v(t, V) = \text{true}$.



Again, we can separate the paths $x_1 = (s, t, s, t, \ldots)$ and $x_2 = (t, s, t, s, \ldots)$.

- $\mathcal{M}, s \models \mathbf{AG}(P \rightarrow \mathbf{F}V)$ holds since x_1 is the only full path that starts from s and $\mathcal{M}, x_1 \models \mathbf{G}(P \rightarrow \mathbf{F}V)$. This is because $\mathcal{M}, x_1^{2k} \models \mathbf{F}V$ (since $\mathcal{M}, x_1^{2k+1} \models V$) holds for all $k \geq 0$ and, additionally, $\mathcal{M}, x_1^{2k+1} \not\models P$ holds for all $k \geq 0$.
  Similarly, $\mathcal{M}, t \models \mathbf{AG}(P \rightarrow \mathbf{F}V)$ holds since x_2 is the only full path that starts from t and $\mathcal{M}, x_2 \models \mathbf{G}(P \rightarrow \mathbf{F}V)$ because $\mathcal{M}, x_2^{2k} \not\models P$ and $\mathcal{M}, x_2^{2k+1} \models \mathbf{F}V$ holds for all $k \geq 0$.
- $\mathcal{M}, s \models \mathbf{AF}\big(P \wedge \mathbf{F}(\neg P \wedge \mathbf{X}P)\big)$ holds since x_1 is the only full path that starts from s and $\mathcal{M}, x_1^0 \models P \wedge \mathbf{F}(\neg P \wedge \mathbf{X}P)$ holds because $\mathcal{M}, x_1^0 \models P$ ($v(s, P) = \text{true}$) and, additionally, $\mathcal{M}, x_1^0 \models \mathbf{F}(\neg P \wedge \mathbf{X}P)$ holds because $\mathcal{M}, x_1^1 \models \neg P \wedge \mathbf{X}P$ (since $v(t, P) = \text{false}$ $\mathcal{M}, (x_1^1)^1 \models P$).
  Similarly, $\mathcal{M}, t \models \mathbf{AF}\big(P \wedge \mathbf{F}(\neg P \wedge \mathbf{X}P)\big)$ holds since x_2 is the only full path that starts from t and because $x_2^1 = x_1 = x_1^0$ and $\mathcal{M}, x_1^0 \models P \wedge \mathbf{F}(\neg P \wedge \mathbf{X}P)$, and (cf. above) $\mathcal{M}, x_2 \models \mathbf{F}\big(P \wedge \mathbf{F}(\neg P \wedge \mathbf{X}P)\big)$.
- $\mathcal{M}, s \models \mathbf{A}(\neg V \mathbf{U} V)$ holds since x_1 is the only full path that starts from s and $\mathcal{M}, x_1 \models \neg V \mathbf{U} V$ because $\mathcal{M}, x_1^1 \models V$ ($v(t, V) = \text{true}$) and $\mathcal{M}, x_1^i \models \neg V$ for all $i < 1$ ($v(s, V) = \text{false}$).
  Since $v(t, V) = \text{true}$, $\mathcal{M}, x \models \neg V \mathbf{U} V$ holds for all full paths x that start from t. Thus $\mathcal{M}, t \models \mathbf{A}(\neg V \mathbf{U} V)$ holds.