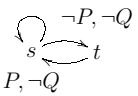
T-79.5101 **Spring 2008**
Advanced Course in Computational Logic
Exercise Session 4
Solutions

1. One possible counterexample is the model $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s\}$, $R = \{\langle s, s\rangle\}$, and $v(s, P) = \text{false}$.

   $\mathcal{M} \models \Box P \rightarrow \Diamond P$ holds (since $\mathcal{M}, s \nVdash \Box P$), and $\mathcal{M}, s \Vdash \Box\neg P$ holds since $\langle s, s\rangle \in R$, $\mathcal{M}, s \Vdash \neg P$ and s has no other successors. Furthermore, $\mathcal{M}, s \Vdash \Box\Box\neg P$ holds. On the other hand, $\mathcal{M}, s \nVdash \Box P$ (and s has no other successor), and hence $\mathcal{M}, s \Vdash \Diamond\Box P$ does not holds. Thus $\mathcal{M}$ is a counterexample.

   (Notice that counterexamples are not unique in general: here other possibilities for counterexamples are, for examples, $\mathcal{M}' = \langle S', R', v'\rangle$, where $S' = \{s', t'\}$, $R' = \{\langle s', t'\rangle, \langle t', s'\rangle\}$ and $v(s', P) = v(t', P) = \text{false}$, and $\mathcal{M}'' = \langle S'', R'', v''\rangle$, $S'' = \{s'', t'', u''\}$, $R'' = \{\langle s'', t''\rangle, \langle t'', u''\rangle, \langle u'', t''\rangle\}$ and $v''(s'', P) = v''(t'', P) = \text{true}$, $v''(u'', P) = \text{false}$, considering the worlds s' and s'', respectively.)

2. $\mathcal{M} = \langle S, R, v\rangle$, where $S = \{s, t\}$, $R = \{\langle s, s\rangle, \langle s, t\rangle, \langle t, s\rangle\}$, $v(s, P) = \text{true}$ and $v(s, Q) = v(t, P) = v(t, Q) = \text{false}$.

   $\mathcal{M}, s \Vdash \Diamond P \vee \Diamond Q$ and $\mathcal{M}, t \Vdash \Diamond P \vee \Diamond Q$ hold (since $\mathcal{M}, s \Vdash P$, $\langle s, s\rangle \in R$ and $\langle t, s\rangle \in R$), and $\mathcal{M}, s \Vdash \neg\Box P$ holds, since $\langle s, t\rangle \in R$ and $\mathcal{M}, t \nVdash P$. However, $\mathcal{M}, s \nVdash \Diamond Q$, since $\mathcal{M}, s' \nVdash Q$ for all $s' \in S$ for which $\langle s, s'\rangle \in R$. Thus $\mathcal{M}$ is a counterexample.

3. Assume that
   $$\Sigma \cup \{P\} \not\models_{\mathbf{L}} \Upsilon \Longrightarrow Q.$$
   Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ such that
   $$\mathcal{M} \models \Sigma \cup \{P\}$$

   and
   $$\exists s \in S: \quad \forall \varphi \in \Upsilon : \mathcal{M}, s \Vdash \varphi, \text{ but } \mathcal{M}, s \nVdash Q.$$
   Especially, $\mathcal{M}, t \Vdash P$ for all $t \in S$, and hence
   $$\mathcal{M}, s \Vdash P \wedge \Box P \wedge \Box\Box P \wedge \Box\Box\Box P.$$
   Since additionally $\mathcal{M} \models \Sigma$ holds we have
   $$\Sigma \not\models_{\mathbf{L}} \Upsilon \Longrightarrow P \wedge \Box P \wedge \Box\Box P \wedge \Box\Box\Box P \rightarrow Q.$$

4. a) Assume that the frame $\mathcal{F} = \langle S, R\rangle$ is transitive and that the formula $\Box P \rightarrow \Box\Box P$ is not valid in the frame. Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ (based on $\mathcal{F}$) and a world $s \in S$ such that $\mathcal{M}, s \nVdash \Box P \rightarrow \Box\Box P$. Now, $\mathcal{M}, s \Vdash \Box P$ but on the other hand $\mathcal{M}, s \nVdash \Box\Box P$. From the latter it follows that there is a world $t \in S$ for which $\langle s, t\rangle \in R$ and $\mathcal{M}, t \nVdash \Box P$. Furthermore, there is a world $u \in S$ for which $\langle t, u\rangle \in R$ and $\mathcal{M}, u \nVdash P$. Since $\langle s, t\rangle \in R$ and $\langle t, u\rangle \in R$ by transitivity of $\mathcal{F}$ we have $\langle s, u\rangle \in R$. Since $\langle s, u\rangle \in R$ and $\mathcal{M}, u \nVdash P$, we have $\mathcal{M}, s \nVdash \Box P$, which is in contradiction with the assumption that $\mathcal{M}, s \Vdash \Box P$ holds. Thus the formula $\Box P \rightarrow \Box\Box P$ is valid in $\mathcal{F}$.

   b) Assume that the frame $\mathcal{F} = \langle S, R\rangle$ is euclidean. Take an arbitrary model $\mathcal{M} = \langle S, R, v\rangle$ based on $\mathcal{F}$ and an arbitrary world $s \in S$ for which $\mathcal{M}, s \Vdash \neg\Box P$ holds. Then $\mathcal{M}, s \nVdash \Box P$, and hence there is a world $t \in S$ such that $\langle s, t\rangle \in R$ and $\mathcal{M}, t \nVdash P$. Assume $\langle s, u\rangle \in R$. Since $\langle s, t\rangle \in R$ and the frame is euclidean, we have $\langle u, t\rangle \in R$. Hence $\mathcal{M}, u \nVdash \Box P$ and $\mathcal{M}, u \Vdash \neg\Box P$. Since u is an arbitrary successor of s, we have $\mathcal{M}, s \Vdash \Box\neg\Box P$, and hence
   $$\mathcal{M}, s \Vdash \neg\Box P \rightarrow \Box\neg\Box P.$$
   Thus $\neg\Box P \rightarrow \Box\neg\Box P$ is valid in $\mathcal{M}$, and $\neg\Box P \rightarrow \Box\neg\Box P$ is valid in $\mathcal{F}$ (since $\mathcal{M}$ was chosen arbitrarily).

5. Assume that $\mathcal{F} = \langle S, R\rangle$ is reflexive and euclidean. If sRt, by reflexivity we have sRs. Since the frame is also euclidean, we have tRs, and thus the frame is symmetric.

   Now assume sRt and tRu. By symmetricity we have tRs. Since the frame is also euclidean, we also have sRu. Thus the frame is transitive.

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Exercise Session 5
Solutions

Axiom K:

$$\text{K:} \quad \Box(P \to Q) \to (\Box P \to \Box Q)$$

Inference rules:

$$\text{MP:} \quad \frac{P, P \to Q}{Q}$$

$$\text{N:} \quad \frac{P}{\Box P}$$

1. a) Assume that the frame $\mathcal{F} = \langle S, R \rangle$ is serial and that the formula $\Box P \to \Diamond P$ is not valid under $\mathcal{F}$. Then there is a model $\mathcal{M} = \langle S, R, v \rangle$ based on $\mathcal{F}$ and a world $s \in S$ such that $\mathcal{M}, s \nVdash \Box P \to \Diamond P$ holds. Hence, $\mathcal{M}, s \Vdash \Box P$ and $\mathcal{M}, s \nVdash \Diamond P$. From $\mathcal{M}, s \nVdash \Diamond P$ it follows that there is no world $t \in S$ such that $\langle s, t \rangle \in R$ and $\mathcal{M}, t \Vdash P$. Furthermore, $\mathcal{F}$ is serial by assumption, and thus there is a world $t \in S$ such that $\langle s, t \rangle \in R$. Hence $\mathcal{M}, s \nVdash \Box P$. A contradiction follows since $\mathcal{M}, s \Vdash \Box P$, and thus the formula $\Box P \to \Diamond P$ is valid under $\mathcal{F}$.

   b) Assume that the frame $\mathcal{F} = \langle S, R \rangle$ is weakly dence and that the formula $\Box\Box P \to \Box P$ is not valid under $\mathcal{F}$. Then there is a model $\mathcal{M} = \langle S, R, v \rangle$ based on $\mathcal{F}$ and a world $s \in S$ such that $\mathcal{M}, s \nVdash \Box\Box P \to \Box P$ holds. Hence $\mathcal{M}, s \Vdash \Box\Box P$ and $\mathcal{M}, s \nVdash \Box P$. From $\mathcal{M}, s \nVdash \Box P$ it follows that there is a world $t \in S$ such that $\langle s, t \rangle \in R$ and $\mathcal{M}, t \nVdash P$. The frame $\mathcal{F}$ is weakly dence by assumption, and thus there is a $u \in S$ such that $\langle s, u \rangle \in R$ and $\langle u, t \rangle \in R$. Since $\langle u, t \rangle \in R$ and $\mathcal{M}, t \nVdash P$, it follows that $\mathcal{M}, u \nVdash \Box P$. Now $\langle s, u \rangle \in R$ and $\mathcal{M}, u \nVdash \Box P$, so $\mathcal{M}, s \nVdash \Box\Box P$ must hold. A contradiction follows since $\mathcal{M}, s \Vdash \Box\Box P$, and thus the formula $\Box\Box P \to \Box P$ is valid under $\mathcal{F}$.

2. a)

1.	$P \to (Q \to P)$	[Tautology]
2.	$\Box\big(P \to (Q \to P)\big)$	[N, 1]
3.	$\Box\big(P \to (Q \to P)\big) \to \big(\Box P \to \Box(Q \to P)\big)$	[K]
4.	$\Box P \to \Box(Q \to P)$	[MP, 2, 3]

   b)

1.	$\Box(P \to Q)$	[GP]
2.	$(P \to Q) \to (\neg Q \to \neg P)$	[Tautology]
3.	$\Box\big((P \to Q) \to (\neg Q \to \neg P)\big)$	[N, 2]
4.	$\Box\big((P \to Q) \to (\neg Q \to \neg P)\big) \to$ $\big(\Box(P \to Q) \to \Box(\neg Q \to \neg P)\big)$	[K]
5.	$\Box(P \to Q) \to \Box(\neg Q \to \neg P)$	[MP, 3, 4]
6.	$\Box(\neg Q \to \neg P)$	[MP, 1, 5]
7.	$\Box(\neg Q \to \neg P) \to (\Box\neg Q \to \Box\neg P)$	[K]
8.	$\Box\neg Q \to \Box\neg P$	[MP, 6, 7]

3. a)

1.	$P \to Q$	[GP]
2.	$\neg Q \to P$	[GP]
3.	$(P \to Q) \to \big((\neg Q \to P) \to Q\big)$	[Tautology]
4.	$(\neg Q \to P) \to Q$	[MP, 1, 3]
5.	Q	[MP, 2, 4]
6.	$\Box Q$	[N, 5]
7.	$\neg Q \vee S$	[LP]
8.	$(\neg Q \vee S) \to (Q \to S)$	[Tautology]
9.	$Q \to S$	[MP, 7, 8]
10.	S	[MP, 5, 9]
11.	$\Box Q \to (S \to \Box Q \wedge S)$	[Tautology]
12.	$S \to \Box Q \wedge S$	[MP, 6, 11]
13.	$\Box Q \wedge S$	[MP, 10, 12]

b)

1.	$Q \to \neg P$	[GP]
2.	$\Box(Q \to \neg P)$	[N, 1]
3.	$\Box(Q \to \neg P) \to (\Box Q \to \Box\neg P)$	[K]
4.	$\Box Q \to \Box\neg P$	[MP, 2, 3]
5.	$\Diamond Q \to \Box Q$	[GP]
6.	$(\Diamond Q \to \Box Q) \to$ $\big((\Box Q \to \Box\neg P) \to (\Diamond Q \to \Box\neg P)\big)$	[Tautology]
7.	$(\Box Q \to \Box\neg P) \to (\Diamond Q \to \Box\neg P)$	[MP, 5, 6]
8.	$\Diamond Q \to \Box\neg P$	[MP, 4, 7]
9.	$(\neg\Box\neg Q \to \Box\neg P) \to (\neg\Box\neg P \to \Box\neg Q)$	[Tautology]
10.	$\neg\Box\neg P \to \Box\neg Q$	[MP, 8, 9]
11.	$\Diamond P$	[LP]
12.	$\Box\neg Q$	[MP, 10, 11]

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Advanced Course in Computational Logic
Exercise Session 6
Solutions

1. a)

1.	$\langle 1\rangle\neg\big(\Box P \to \Box(Q \to P)\big)$	
2.	$\langle 1\rangle\Box P$	(1)
3.	$\langle 1\rangle\neg\Box(Q \to P)$	(1)
4.	$\langle 1,2\rangle\neg(Q \to P)$	(3)
5.	$\langle 1,2\rangle Q$	(4)
6.	$\langle 1,2\rangle\neg P$	(4)
7.	$\langle 1,2\rangle P$	(2)
	$\otimes$	

b)

1.	$\langle 1\rangle\neg\big(\Box(P \to Q) \to (\neg\Box\neg P \to \neg\Box\neg Q)\big)$				
2.	$\langle 1\rangle\Box(P \to Q)$	(1)			
3.	$\langle 1\rangle\neg(\neg\Box\neg P \to \neg\Box\neg Q)$	(1)			
4.	$\langle 1\rangle\neg\Box\neg P$	(3)			
5.	$\langle 1\rangle\neg\neg\Box\neg Q$	(3)			
6.	$\langle 1\rangle\Box\neg Q$	(5)			
7.	$\langle 1,2\rangle\neg\neg P$	(4)			
8.	$\langle 1,2\rangle P$	(7)			
9.	$\langle 1,2\rangle\neg Q$	(6)			
10.	$\langle 1,2\rangle P \to Q$	(2)			
11.	$\langle 1,2\rangle\neg P$ (10)		12.	$\langle 1,2\rangle Q$ (10)	
	$\otimes$			$\otimes$	

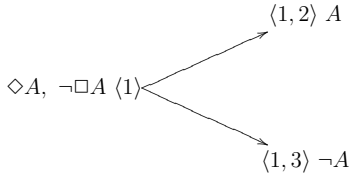
c)

1.	$\langle 1\rangle\neg\big((\Box P \wedge \Box Q) \to \Box(P \wedge Q)\big)$	
2.	$\langle 1\rangle\Box P \wedge \Box Q$	(1)
3.	$\langle 1\rangle\neg\Box(P \wedge Q)$	(1)
4.	$\langle 1\rangle\Box P$	(2)
5.	$\langle 1\rangle\Box Q$	(2)
6.	$\langle 1,2\rangle\neg(P \wedge Q)$	(3)
7.	$\langle 1,2\rangle P$	(4)
8.	$\langle 1,2\rangle Q$	(5)
9.	$\langle 1,2\rangle\neg P$ (6)	10. $\langle 1,2\rangle\neg Q$ (6)
	$\otimes$	$\otimes$

2. a)

1.	$\langle 1\rangle\neg(\Diamond A \to \Box A)$	
2.	$\langle 1\rangle\Diamond A$	(1)
3.	$\langle 1\rangle\neg\Box A$	(1)
4.	$\langle 1,2\rangle\neg\neg A$	(2; $\Diamond$ stands for $\neg\Box\neg$)
5.	$\langle 1,2\rangle A$	(4)
6.	$\langle 1,3\rangle\neg A$	(3)

Not **K**-valid.



b)

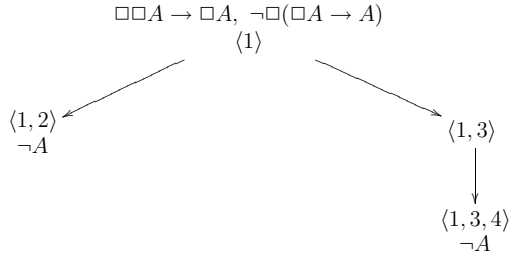
1.	$\langle 1\rangle\neg(\Diamond\Box A \vee \Box\Diamond\neg A)$	
2.	$\langle 1\rangle\neg\Diamond\Box A$	(1)
3.	$\langle 1\rangle\neg\Box\Diamond\neg A$	(1)
4.	$\langle 1\rangle\Box\neg\Box A$	(2)
5.	$\langle 1,2\rangle\neg\Diamond\neg A$	(3)
6.	$\langle 1,2\rangle\neg\Box A$	(4)
7.	$\langle 1,2\rangle\Box\neg\neg A$	(5)
8.	$\langle 1,2,3\rangle\neg A$	(6)
9.	$\langle 1,2,3\rangle\neg\neg A$	(7)
	$\otimes$	

The formula is **K**-valid.

c)

1.	$\langle 1\rangle\neg\big((\Box\Box A \to \Box A) \to \Box(\Box A \to A)\big)$				
2.	$\langle 1\rangle\Box\Box A \to \Box A$	(1)			
3.	$\langle 1\rangle\neg\Box(\Box A \to A)$	(1)			
4.	$\langle 1,2\rangle\neg(\Box A \to A)$	(3)			
5.	$\langle 1,2\rangle\Box A$	(4)			
6.	$\langle 1,2\rangle\neg A$	(4)			
7.	$\langle 1\rangle\neg\Box\Box A$	(2)	8.	$\langle 1\rangle\Box A$	(2)
9.	$\langle 1,3\rangle\neg\Box A$	(7)	11.	$\langle 1,2\rangle A$	(8)
10.	$\langle 1,3,4\rangle\neg A$	(9)		$\otimes$	

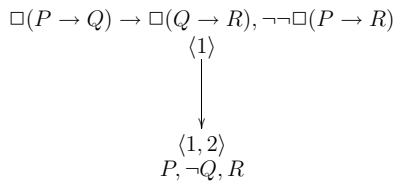
Not **K**-valid.



3. a)

1.	$\langle 1\rangle\neg\Big(\big(\Box(P \to Q) \to \Box(Q \to R)\big) \to \neg\Box(P \to R)\Big)$				
2.	$\langle 1\rangle\Box(P \to Q) \to \Box(Q \to R)$	(1)			
3.	$\langle 1\rangle\neg\neg\Box(P \to R)$	(1)			
4.	$\langle 1\rangle\Box(P \to R)$	(3)			
5.	$\langle 1\rangle\neg\Box(P \to Q)$	(2)	6.	$\langle 1\rangle\Box(Q \to R)$	(2)
7.	$\langle 1,2\rangle\neg(P \to Q)$	(5)			
8.	$\langle 1,2\rangle P$	(7)			
9.	$\langle 1,2\rangle\neg Q$	(7)			
10.	$\langle 1,2\rangle P \to R$	(4)			
11.	$\langle 1,2\rangle\neg P$ (10)		12.	$\langle 1,2\rangle R$	(10)
	$\otimes$				

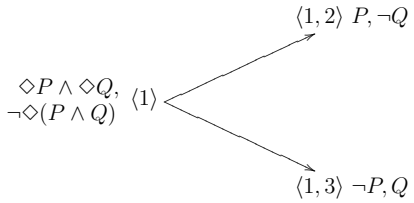
Not **K**-valid.



b)

1.	$\langle 1\rangle\neg\big((\Diamond P \wedge \Diamond Q) \rightarrow \Diamond(P \wedge Q)\big)$				
2.	$\langle 1\rangle\Diamond P \wedge \Diamond Q$			(1)	
3.	$\langle 1\rangle\neg\Diamond(P \wedge Q)$			(1)	
4.	$\langle 1\rangle\Diamond P$			(2)	
5.	$\langle 1\rangle\Diamond Q$			(2)	
6.	$\langle 1\rangle\Box\neg(P \wedge Q)$			(3)	
7.	$\langle 1,2\rangle P$			(4)	
8.	$\langle 1,2\rangle\neg(P \wedge Q)$			(6)	
9.	$\langle 1,2\rangle\neg P$	(8)	10.	$\langle 1,2\rangle\neg Q$	(8)
	$\otimes$		11.	$\langle 1,3\rangle Q$	(5)
			12.	$\langle 1,3\rangle\neg(P \wedge Q)$	(6)
			13.	$\langle 1,3\rangle\neg P$	(11)
			13.	$\langle 1,3\rangle\neg Q$	(12)
				$\otimes$	

Not **K**-valid.



c)

1.	$\langle 1\rangle\neg\big(\Box(P \wedge Q) \rightarrow (\Box P \wedge \Box Q)\big)$				
2.	$\langle 1\rangle\Box(P \wedge Q)$				(1)
3.	$\langle 1\rangle\neg(\Box P \wedge \Box Q)$				(1)
4.	$\langle 1\rangle\neg\Box P$	(3)	5.	$\langle 1\rangle\neg\Box Q$	(3)
6.	$\langle 1,2\rangle\neg P$	(4)	10.	$\langle 1,2\rangle\neg Q$	(5)
7.	$\langle 1,2\rangle P \wedge Q$	(2)	11.	$\langle 1,2\rangle P \wedge Q$	(2)
8.	$\langle 1,2\rangle P$	(7)	12.	$\langle 1,2\rangle P$	(11)
9.	$\langle 1,2\rangle Q$	(7)	13.	$\langle 1,2\rangle Q$	(11)
	$\otimes$			$\otimes$	

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Advanced Course in Computational Logic
Exercise Session 7
Solutions

1. a)

1.	$\langle 1\rangle\neg\big(\Diamond(P \vee \Diamond\Box\Box P) \rightarrow \Diamond P\big)$				
2.	$\langle 1\rangle\Diamond(P \vee \Diamond\Box\Box P)$				(1)
3.	$\langle 1\rangle\neg\Diamond P$				(1)
4.	$\langle 1,2\rangle P \vee \Diamond\Box\Box P$				(2)
5.	$\langle 1,2\rangle\neg P$				(3)
6.	$\langle 1,2\rangle P$	(4)	7.	$\langle 1,2\rangle\Diamond\Box\Box P$	(4)
	$\otimes$		8.	$\langle 1,2,3\rangle\Box\Box P$	(7)
			9.	$\langle 1,2,3\rangle\Box P$	(8) (reflexivity)
			10.	$\langle 1,2,3\rangle P$	(9) (reflexivity)
			11.	$\langle 1,2,3\rangle\neg P$	(3) (transitivity)
				$\otimes$	

b)

1.	$\langle 1\rangle\neg\Diamond\big(\Diamond P \rightarrow \Box(\Diamond P \vee P)\big)$			
2.	$\langle 1\rangle\neg\big(\Diamond P \rightarrow \Box(\Diamond P \vee P)\big)$		(1)	(reflexivity)
3.	$\langle 1\rangle\Diamond P$		(2)	
4.	$\langle 1\rangle\neg\Box(\Diamond P \vee P)$		(2)	
5.	$\langle 1,2\rangle\neg(\Diamond P \vee P)$		(4)	
6.	$\langle 1,2\rangle\neg\Diamond P$		(5)	
7.	$\langle 1,2\rangle\neg P$		(5)	
8.	$\langle 1,2\rangle\neg\big(\Diamond P \rightarrow \Box(\Diamond P \vee P)\big)$		(1)	
9.	$\langle 1,2\rangle\Diamond P$		(8)	
10.	$\langle 1,2\rangle\neg\Box(\Diamond P \vee P)$		(8)	
11.	$\langle 1,2,3\rangle P$		(9)	
12.	$\langle 1,2,3\rangle\neg P$		(6)	
	$\otimes$			

c)

1.	$\langle 1\rangle\neg\Box\big(\Box(\Box P\wedge Q)\rightarrow\Diamond\Box\Diamond\Diamond(P\vee Q)\big)$	
2.	$\langle 1,2\rangle\neg\big(\Box(\Box P\wedge Q)\rightarrow\Diamond\Box\Diamond\Diamond(P\vee Q)\big)$	(1)
3.	$\langle 1,2\rangle\Box(\Box P\wedge Q)$	(2)
4.	$\langle 1,2\rangle\neg\Diamond\Box\Diamond\Diamond(P\vee Q)$	(2)
5.	$\langle 1\rangle\Box P\wedge Q$	(3) (symmetricity)
6.	$\langle 1\rangle\Box P$	(5)
7.	$\langle 1\rangle Q$	(5)
8.	$\langle 1\rangle\neg\Box\Diamond\Diamond(P\vee Q)$	(4) (symmetricity)
9.	$\langle 1,3\rangle\neg\Diamond\Diamond(P\vee Q)$	(8)
10.	$\langle 1\rangle\neg\Diamond(P\vee Q)$	(9) (symmetricity)
11.	$\langle 1,3\rangle\neg(P\vee Q)$	(10)
12.	$\langle 1,3\rangle\neg P$	(11)
13.	$\langle 1,3\rangle\neg Q$	(11)
14.	$\langle 1,3\rangle P$	(6)
	$\otimes$	

d)

1.	$\langle 1\rangle\neg\Big(\Box P\rightarrow\Diamond\big((P\rightarrow\Box Q)\rightarrow Q\big)\Big)$	
2.	$\langle 1\rangle\Box P$	(1)
3.	$\langle 1\rangle\neg\Diamond\big((P\rightarrow\Box Q)\rightarrow Q\big)$	(1)
4.	$\langle 1,2\rangle P$	(2) (serial)
5.	$\langle 1,2\rangle\neg\big((P\rightarrow\Box Q)\rightarrow Q\big)$	(3)
6.	$\langle 1,2\rangle P\rightarrow\Box Q$	(5)
7.	$\langle 1,2\rangle\neg Q$	(5)
8.	$\langle 1,2\rangle\neg P$ (6) $\otimes$	
9.	$\langle 1,2\rangle\Box Q$	(6)
10.	$\langle 1,2,3\rangle Q$	(9) (serial)
11.	$\langle 1,2,3\rangle\neg\big((P\rightarrow\Box Q)\rightarrow Q\big)$	(3) (transitivity)
12.	$\langle 1,2,3\rangle P\rightarrow\Box Q$	(11)
13.	$\langle 1,2,3\rangle\neg Q$	(11)
	$\otimes$	

e)

1.	$1\ \neg\Diamond(\Box\Diamond\Box P\rightarrow\Box P)$	
2.	$1\ \neg(\Box\Diamond\Box P\rightarrow\Box P)$	(1)
3.	$1\ \Box\Diamond\Box P$	(2)
4.	$1\ \neg\Box P$	(2)
5.	$2\ \neg P$	(4)
6.	$2\ \Diamond\Box P$	(3)
7.	$3\ \Box P$	(6)
8.	$2\ P$	(7)
	$\otimes$	

2. Systematic **K**-tableau:

1.	$\langle 1\rangle\neg(\Diamond P\rightarrow\Diamond\Box P)$	
2.	$\langle 1\rangle\Diamond P$	(1)
3.	$\langle 1\rangle\neg\Diamond\Box P$	(1)
4.	$\langle 1,2\rangle P$	(2)
5.	$\langle 1,2\rangle\neg\Box P$	(3)
6.	$\langle 1\rangle\neg\Diamond\Box P$	(3)
7.	$\langle 1,2,3\rangle\neg P$	(5)

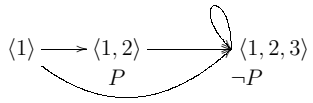
Not **K**-valid. $\langle 1\rangle \longrightarrow \langle 1,2\rangle \longrightarrow \langle 1,2,3\rangle$ with P and $\neg P$.

Systematic **K4**-tableau:

1.	$\langle 1\rangle\neg(\Diamond P\rightarrow\Diamond\Box P)$	
2.	$\langle 1\rangle\Diamond P$	(1)
3.	$\langle 1\rangle\neg\Diamond\Box P$	(1)
4.	$\langle 1,2\rangle P$	(2)
5.	$\langle 1,2\rangle\neg\Box P$	(3)
6.	$\langle 1\rangle\neg\Diamond\Box P$	(3)
7.	$\langle 1,2,3\rangle\neg P$	(5)
8.	$\langle 1,2,3\rangle\neg\Box P$	(6) (transitivity)
9.	$\langle 1\rangle\neg\Diamond\Box P$	(6)
10.	$\langle 1,2,3,4\rangle\neg P$	(8)
11.	$\langle 1,2,3,4\rangle\neg\Box P$	(9) (transitivity)
12.	$\langle 1\rangle\neg\Diamond\Box P$	(9)
13.	$\langle 1,2,3,4,5\rangle\neg P$	(11)
14.	$\langle 1,2,3,4,5\rangle\neg\Box P$	(12) (transitivity)
15.	$\langle 1\rangle\neg\Diamond\Box P$	(12)
	$\vdots$	

We cannot obtain a complete tableau since an infinite branch is generated into the systematic **K4**-tableau. Since this infinite branch is open, it follows that the formula $\Diamond P \to \Diamond\Box P$ is not **K4**-valid.

Notice that the formulas $\neg P$ and $\neg\Box P$ appear repeatedly in the prefices $\langle 1,2,3\rangle$, $\langle 1,2,3,4\rangle$, and $\langle 1,2,3,4,5\rangle$. Therefore exactly the same formulas hold in the worlds corresponding to these prefices in a countermodel. We attempt to construct a finite countermodel by seeing all such worlds as one. We will then check whether the model that follows is really a countermodel for the claim that the formula given in the exercise is **K4**-valid. When we at the same time assure that the model is based on a transitive frame, we end up with the model



The formula $\Diamond P$ is true in world $\langle 1\rangle$, but $\Diamond\Box P$ is not. Therefore this model is a countermodel for the claim that the formula given in the exercise is **K4**-valid.

3.

1.	$\langle 1\rangle\neg\big((\Diamond P \to \Diamond\Diamond P) \wedge \neg P\big)$				
2.	$\langle 1\rangle\neg(\Diamond P \to \Diamond\Diamond P)$	(1)	3.	$\langle 1\rangle\neg\neg P$	(1)
6.	$\langle 1\rangle\Diamond P$	(2)	4.	$\langle 1\rangle P$	(3)
7.	$\langle 1\rangle\neg\Diamond\Diamond P$	(2)	5.	$\langle 1\rangle\neg P$	(LP)
8.	$\langle 1,2\rangle P$	(6)		$\otimes$	
9.	$\langle 1,2\rangle\neg\Diamond P$	(7)			
10.	$\langle 1,2\rangle\Box\neg P \to \neg P$	(GP)			
11.	$\langle 1,2\rangle\neg\Box\neg P$	(10)	12.	$\langle 1,2\rangle\neg P$	(10)
13.	$\langle 1,2,3\rangle\neg\neg P$	(11)		$\otimes$	
14.	$\langle 1,2,3\rangle P$	(12)			
15.	$\langle 1,2,3\rangle\neg P$	(9)			
	$\otimes$				