1. $\mathcal{M}$:

![Diagram of $\mathcal{M}$]

a) $\mathcal{M}, a \not\models A(P \lor Q)$, since (for example) the full path $(a, b, d, d, \ldots)$ starts from state $a$ but does not go through any state $s \in S$ in which $\mathcal{M}, s \models Q$ holds.

b) A full path in a model is F-fair if and only if every $\varphi \in F$ is infinitely often true on the path. Since $\{s \in S | \mathcal{M}, s \models R\} = \{f\}$, it follows that all F-fair paths in $\mathcal{M}$ must visit state $f$ infinitely often. Since $f$ is not reachable from either of the states $c$ and $d$, there is no F-fair path that visits these two states. Hence every F-fair path in $\mathcal{M}$ can be represented as $(a, b, e, \ldots, f, a, b, e, \ldots, f, a, b, e, \ldots)$ where $n_1$, $n_2$, $n_3$, \ldots are (finite) positive integers. Especially, since $n_1$ is finite and, furthermore, $\mathcal{M}, a \models P$, $\mathcal{M}, b \models P$, $\mathcal{M}, e \models P$ and $\mathcal{M}, f \models Q$ hold, it follows that $\mathcal{M}, a \models A(P \lor Q)$ holds.

c) $\mathcal{M}, a \models EG P$ holds since there is the full path $(a, b, e, e, \ldots)$ and for each state $s \in \{a, b, e\}$ we have $\mathcal{M}, s \models P$.

d) Notice that the path $(a, b, e, e, \ldots)$ is the only full path in which $P$ holds and which starts from $a$. However, this path is not F-fair since it does not visit the state $f$. Hence $\mathcal{M}, a \not\models EG P$.

2. $\mathcal{M}$:

![Diagram of $\mathcal{M}$]

We sort the subformulas of $AXE((P \rightarrow Q) \lor (P \land Q))$ so that the truth value of each subformula can be iteratively determined when the truth values of the preceding subformulas are known. One such order is $P, Q, P \rightarrow Q, P \land Q, E((P \rightarrow Q) \lor (P \land Q)), AXE((P \rightarrow Q) \lor (P \land Q))$.

The truth values of the formulas $P$ and $Q$ are given directly by the valuation $v$. These in turn allow us to evaluate $P \rightarrow Q$ in each of the states of the model:

![Diagram of $\mathcal{M}$]

Next we evaluate $E((P \rightarrow Q) \lor (P \land Q))$ using the algorithm CheckEU in the lecture notes. We start from the set of states in which $P \land Q$ is true ($\{b\}$) and mark $E((P \rightarrow Q) \lor (P \land Q))$ as true in those states. Then we collect all states $s \in S$ such that $s \not\models Q$ (disregarding those states in which $E((P \rightarrow Q) \lor (P \land Q))$ is already marked as true). We arrive at the set of states $\{d, e\}$, and hence mark $E((P \rightarrow Q) \lor (P \land Q))$ as true in these states. Repeat this for the predecessors of $d$ and $e$, and again for the predecessors of the states we arrive at, iteratively until we do not arrive at any new states. This procedure can be described as follows.
<table>
<thead>
<tr>
<th>Round</th>
<th>Visited states</th>
<th>Considered states</th>
<th>New states</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{b}</td>
<td>{b}</td>
<td>{d, e}</td>
</tr>
<tr>
<td>2</td>
<td>{b, d, e}</td>
<td>{d, e}</td>
<td>{c}</td>
</tr>
<tr>
<td>3</td>
<td>{b, c, d, e}</td>
<td>{c}</td>
<td>\emptyset</td>
</tr>
</tbody>
</table>

We now know that $E((P \rightarrow Q)U(P \land Q))$ is true precisely in the set of states $\{b, c, d, e\}$.

Then, sort the subformulas into a suitable order.

\[ P, Q, \neg P, EG\rightarrow P, EGE\rightarrow P, \neg EGE\rightarrow P \]
\[ E(TU)P, \neg E(TU)P, \neg E(TU) \land EG\rightarrow P \]
\[ E((EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \]
\[ E((EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \]
\[ \neg (Q \rightarrow \neg E(EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \land \neg EGE\rightarrow P \]
\[ E(TU)(Q \rightarrow \neg E(EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \land \neg EGE\rightarrow P \]
\[ E(TU)(Q \rightarrow \neg E(EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \land \neg EGE\rightarrow P \]

Since $P$ is true in the states $a$ and $c$, $\neg P$ is true in the states $\{b, d, e\}$. Now we can apply the algorithm CheckEG to evaluate $EG\rightarrow P$. First, construct the restriction $M'$ based on the states in which $\neg P$ is true.

Then, find the non-trivial strongly connected components (SCCs) of $M'$, and mark $EG\rightarrow P$ as true in all states belonging to one of the non-trivial SCCs.

Now, similarly as in the algorithm CheckEU, iteratively collect all the states that are predecessors of at least one state in the non-trivial SCCs of $M'$. Here the only non-trivial SCC of $M'$ is $\{b, d\}$. Since the state $c$ is not a predecessor of $b$ or $d$, the CheckEG algorithm terminates immediately.

For the evaluation we employ the CheckEG and CheckEU algorithms. First, we have to express the formula using the operators $EU$ and $EG$:

\[ AG(Q \rightarrow E(AFPUAFP)) \]
\[ = AG(Q \rightarrow E((EFPUA)P)) \]
\[ = AG(Q \rightarrow E((EFPUA)P)) \]
\[ = AG(Q \rightarrow E((EFPUA)P)) \]
\[ = E((Q \rightarrow E(EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \land \neg EGE\rightarrow P) \]
\[ = E((Q \rightarrow E(EG\rightarrow P)U(E(TU) \land EG\rightarrow P)) \land \neg EGE\rightarrow P) \]

Hence we arrive at

\[ \neg P, Q, \neg P, \neg P \]
\[ \neg P \]
\[ \neg P, Q, \neg P \]
\[ \neg P, Q,\neg P \]
Now determine the states in which the subformula $\text{E}\text{G}\text{E}\text{G} \neg P$ is true using $\text{CheckEG}$. We consider the restriction $M''$ based on those states in which $\text{E}\text{G} \neg P$ is true.

$$\neg P, Q, \text{E}\text{G} \neg P \quad b$$

$$\neg P, \neg Q, \text{E}\text{G} \neg P \quad d$$

The only non-trivial SCC of $M''$ is $\{b, d\}$. Again, $\text{CheckEG}$ terminates immediately.

<table>
<thead>
<tr>
<th>Round</th>
<th>Collected states</th>
<th>Considered states</th>
<th>New states</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${b, d}$</td>
<td>${b, d}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

We now know that the formula $\text{E}\text{G}\text{E}\text{G} \neg P$ is true in the states $b$ and $d$, and hence $\neg \text{E}\text{G}\text{E}\text{G} \neg P$ is true in the states $a$, $c$, and $e$.

$$\neg P, Q, \text{E}\text{G} \neg P \quad b$$

$$\neg P, \neg Q, \text{E}\text{G} \neg P \quad c$$

$$\neg P, \neg Q, \neg \text{E}\text{G}\text{E}\text{G} \neg P \quad d$$

Now evaluate the subformula $\text{E}(\top U P)$ using $\text{CheckEU}$:

<table>
<thead>
<tr>
<th>Round</th>
<th>Collected states</th>
<th>Considered states</th>
<th>New states</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${a, c}$</td>
<td>${a, c}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>2</td>
<td>${a, c, e}$</td>
<td>${c}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>3</td>
<td>${a, c, d, e}$</td>
<td>${d}$</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>4</td>
<td>${a, b, c, d, e}$</td>
<td>${b}$</td>
<td>$\emptyset$</td>
</tr>
</tbody>
</table>

Thus, the formula $\text{E}(\top U P)$ is true in all states of $M$, and therefore $\neg \text{E}(\top U P)$ is false in all states. It follows that the conjunction $\neg \text{E}(\top U P) \land \text{E} \neg P$ is also false in all the states.

Then we again use $\text{CheckEU}$ for considering $\text{E}(\text{E} \neg P \lor (\neg \text{E}(\top U P) \land \neg \text{E} \neg P))$. The algorithm terminates immediately, since the set of states from which we start from is empty by the above.

Thus the formula $\neg \text{E}(\text{E} \neg P \lor (\neg \text{E}(\top U P) \land \neg \text{E} \neg P))$ is true in all the states of $M$. Since we already know that $\neg \text{E}\text{G}\text{E}\text{G} \neg P$ is true in the states $a, c$, and $e$, we can evaluate the conjunction

$$\varphi = \neg \text{E}(\text{E} \neg P \lor (\neg \text{E}(\top U P) \land \text{E} \neg P))$$

in each state of the model:

(For clarity, only those subformulas are visible which are still needed.)

Now the subformula $Q \rightarrow \varphi$:

The subformula $\neg(Q \rightarrow \varphi)$:

The subformula $\text{E}(\top U (Q \rightarrow \varphi))$ using $\text{CheckEU}$:
Finally, we arrive at:

\[
\begin{array}{c}
\text{a} \\
\downarrow \\
\text{b}
\end{array}
\quad
\begin{array}{c}
\text{c} \\
\downarrow \\
\text{d}
\end{array}
\quad
\begin{array}{c}
\text{e}
\end{array}
\]

The closure of \(X(\neg PUQ)\):

\[
\text{CL}(X(\neg PUQ)) = \{X(\neg PUQ), \neg X(\neg PUQ), \neg PUQ, X(\neg PUQ), X(\neg PUQ), \neg P, Q, X(\neg PUQ), P, \neg Q\}
\]

Now construct the atoms. Since \(v(a, P) = v(a, Q) = \text{false}\), we can construct the following atomic tableau:

\[
\begin{array}{c}
\top \\
\downarrow \\
\neg P
\end{array}
\quad
\begin{array}{c}
\neg PUQ \\
\downarrow \\
X(\neg PUQ)
\end{array}
\quad
\begin{array}{c}
\neg PUQ \\
\downarrow \\
\neg X(\neg PUQ)
\end{array}
\]

The open branches in the tableau give us the sets of formulas

\[
K_1 = \{\top, \neg P, \neg Q, \neg PUQ, X(\neg PUQ), X(\neg PUQ)\}
\]

\[
K_2 = \{\top, \neg P, \neg Q, \neg PUQ, X(\neg PUQ), X(\neg PUQ)\}
\]

Hence for state \(a\) we obtain the atoms \((a, K_1)\) and \((a, K_2)\).

Since the valuation for the formulas \(P\) and \(Q\) in state \(c\) is identical to the valuation in state \(a\), for state \(c\) we obtain the atoms \((c, K_1)\) and \((c, K_2)\).
Now consider state $b$.

The open branches in the tableau give us the sets of formulas

$$K_3 = \{ \top, P, Q, \neg P \vee Q, X(\neg P \vee Q), \neg X(\neg P \vee Q) \}$$

$$K_4 = \{ \top, P, Q, \neg P \vee Q, \neg X(\neg P \vee Q), X(\neg P \vee Q) \}$$

Thus for $b$ we have the atoms $(b, K_3)$ and $(b, K_4)$.

Then consider state $d$.

Now we arrive at the sets of formulas

$$K_5 = \{ \top, P, Q, \neg P \vee Q, X(\neg P \vee Q), \neg X(\neg P \vee Q) \}$$

$$K_6 = \{ \top, P, Q, \neg P \vee Q, \neg X(\neg P \vee Q), X(\neg P \vee Q) \}$$

(Two open branches give us $K_6$) Thus for $d$ we have the atoms $(d, K_5)$ and $(d, K_6)$.

Then, we construct a graph $G = (N, E)$. The set of nodes $N$ is the set of atoms we have just obtained, that is,

$$N = \{ (a, K_1), (a, K_2), (b, K_4), (b, K_5), (c, K_1), (c, K_2), (d, K_5), (d, K_6) \}.$$

The set of edges $E$ is defined as follows: there is an edge from the atom $(s, K)$ to atom $(s', K')$ if and only if

(a) $sR_e$ (in model $M$), and
(b) for each formula $X\varphi \in CL(X(\neg P \vee Q))$ we have $X\varphi \in K$ if and only if $\varphi \in K'$.

To check condition (b) we can first form a “compatibility relation” over the atoms:

In the table, there is a tick in the $j$th column of the $i$th row if and only if the condition $X\varphi \in K_i$ if and only if $\varphi \in K_j$ holds for each $X\varphi \in CL(X(\neg P \vee Q))$. For example, this is fulfilled by $(K_1, K_3)$ since $X(\neg P \vee Q) \in K_1$ is the only formula in $K_1$ of the form $X\varphi$ and we have $\neg P \vee Q \in K_3$.

The conditions (a) and (b) can now be checked using $R$ and this table. As an example, consider the atom $(b, K_3)$. Now any atom in \{$(a, K_1), (a, K_2), (d, K_5), (d, K_6)$\} together with $(b, K_3)$ fulfills condition (a). However, since condition (b) does not hold for any of $(K_3, K_2), (K_2, K_3), (K_3, K_5), (K_5, K_6)$, the only edge from $(b, K_3)$ is $(b, K_3)$.

In the end we arrive at the graph $G$:

In order to evaluate $EX(\neg P \vee Q)$ in state $a$, we check whether there is a path $x$ in $G$ fulfilling the following conditions: (i) $x$ begins at one of the atoms $(a, K)$ $(K \in \{ K_1, \ldots, K_6 \})$, (ii) $X(\neg P \vee Q)$ is in $K$, and (iii) $x$ leads to a self-fulfilling non-trivial SCC of $G$.

The only non-trivial SCC of $G$ is

$$C = \{ (a, K_1), (a, K_2), (b, K_4), (c, K_2), (d, K_5), (d, K_6) \}.$$
This SCC is also self-fulfilling, since the only formula of the form $\phi \mathcal{U} \psi$
appearing in the atoms of $C$ is $\neg P \mathcal{U} Q$, and $C$ includes an atom for
which $Q \in K_3$ holds ($a, K_3$), for example.
Since $(a, K_1) \in C$, the self-fulfilling non-trivial SCC $C$ is reachable
from $(a, K_1)$. Thus, since $X(\neg P \mathcal{U} Q) \in K_1$, we know that $\mathcal{M}, a \models EX(\neg P \mathcal{U} Q)$ holds.

2. $\mathcal{M}$:

\[
\begin{array}{c}
\begin{array}{c}
\text{P} \\
\text{↑}
\end{array}
\end{array}
\]

$\mathcal{M}, a \models AFG \phi$ iff $\mathcal{M}, a \models \neg E \neg FG \phi$ iff $\mathcal{M}, a \not\models EFG \phi$. Hence we will investigate whether $\mathcal{M}, a \models EFG \phi$ holds. First, rewrite $\neg FG \phi$
using only the temporal connectives $X$ and $\mathcal{U}$.

\[
\neg FG \phi \equiv \neg F \neg G \phi \\
\equiv \neg F(\neg (\mathcal{U} \neg (\mathcal{U} \neg P)))
\]

The closure of $\neg (\mathcal{U} \neg (\mathcal{U} \neg P))$:

$CL(\neg (\mathcal{U} \neg (\mathcal{U} \neg P))) = \{ \neg (\mathcal{U} \neg (\mathcal{U} \neg P)), \mathcal{U} \neg (\mathcal{U} \neg P), \\
T, \neg (\mathcal{U} \neg P), X(\mathcal{U} \neg (\mathcal{U} \neg P)), \neg T, \\
\mathcal{U} \neg (\mathcal{U} \neg P), X(\mathcal{U} \neg (\mathcal{U} \neg P)), \neg P, \\
\mathcal{U} \neg (\mathcal{U} \neg P), X(\mathcal{U} \neg (\mathcal{U} \neg P)), \neg X(\mathcal{U} \neg (\mathcal{U} \neg P)), \\
X(\mathcal{U} \neg (\mathcal{U} \neg P)), X(\mathcal{U} \neg (\mathcal{U} \neg P)) \}$

Construct the atoms. Since $v(a, P) = \text{false}$, for state $a$ we can construct
the atomic tableau

\[
\begin{array}{c}
\begin{array}{c}
\text{T} \\
\text{P}
\end{array}
\end{array}
\]

where the branch $T_1$ is

\[
\begin{array}{c}
\begin{array}{c}
\text{T} \\
\text{P}
\end{array}
\end{array}
\]

From the open branches we obtain

$K_1 = \{ T, \neg P, T \mathcal{U} P, T \mathcal{U} (T \mathcal{U} \neg P), X(T \mathcal{U} (T \mathcal{U} \neg P)), \\
\neg X(T \mathcal{U} (T \mathcal{U} \neg P)), T \mathcal{U} (T \mathcal{U} \neg P) \}$

$K_2 = \{ T, \neg P, T \mathcal{U} P, T \mathcal{U} (T \mathcal{U} \neg P), X(T \mathcal{U} (T \mathcal{U} \neg P)), \\
\neg X(T \mathcal{U} (T \mathcal{U} \neg P)), T \mathcal{U} (T \mathcal{U} \neg P) \}$

$K_3 = \{ T, \neg P, T \mathcal{U} P, T \mathcal{U} (T \mathcal{U} \neg P), X(T \mathcal{U} (T \mathcal{U} \neg P)), \\
\neg X(T \mathcal{U} (T \mathcal{U} \neg P)), T \mathcal{U} (T \mathcal{U} \neg P) \}$

$K_4 = \{ T, \neg P, T \mathcal{U} P, T \mathcal{U} (T \mathcal{U} \neg P), X(T \mathcal{U} (T \mathcal{U} \neg P)), \\
\neg X(T \mathcal{U} (T \mathcal{U} \neg P)), T \mathcal{U} (T \mathcal{U} \neg P) \}$

$K_5 = \{ T, \neg P, T \mathcal{U} P, X(T \mathcal{U} (T \mathcal{U} \neg P)), X(T \mathcal{U} (T \mathcal{U} \neg P)), \\
\neg X(T \mathcal{U} (T \mathcal{U} \neg P)), T \mathcal{U} (T \mathcal{U} \neg P) \}$

$K_6 = \{ T, \neg P, T \mathcal{U} P, X(T \mathcal{U} (T \mathcal{U} \neg P)), X(T \mathcal{U} (T \mathcal{U} \neg P)), \\
\neg X(T \mathcal{U} (T \mathcal{U} \neg P)), T \mathcal{U} (T \mathcal{U} \neg P) \}$

Thus we obtain the atoms $(a, K_1), (a, K_2), (a, K_3), (a, K_4), (a, K_5)$, and
$(a, K_6)$.
Since \( v(b, P) = v(c, P) \) is true, for \( b \) and \( c \) we can construct the atomic tableau

\[
\begin{align*}
& T
\end{align*}
\]

\[
\begin{align*}
& P \quad \neg P \\
& \otimes
\end{align*}
\]

\[
\begin{align*}
& \neg X(T\cup\neg P) \quad X(T\cup\neg P) \\
& \otimes
\end{align*}
\]

\[
\begin{align*}
& \neg \neg X(T\cup\neg P) \\
& \otimes
\end{align*}
\]

where the branch \( T_3 \) is

\[
\begin{align*}
& T
\end{align*}
\]

\[
\begin{align*}
& P \quad \neg P \\
& \otimes
\end{align*}
\]

\[
\begin{align*}
& \neg X(T\cup\neg P) \quad X(T\cup\neg P) \\
& \otimes
\end{align*}
\]

and the branch \( T_4 \)

\[
\begin{align*}
& T
\end{align*}
\]

\[
\begin{align*}
& P \quad \neg P \\
& \otimes
\end{align*}
\]

\[
\begin{align*}
& \neg X(T\cup\neg P) \quad X(T\cup\neg P) \\
& \otimes
\end{align*}
\]

From the open branches we obtain

\[
\begin{align*}
K_7 &= \{ T, P, \neg \neg \neg X(TU\neg P), \neg X(TU\neg P), \neg X((TU\neg P), \neg X(TU\neg P)) \} \\
K_8 &= \{ T, P, \neg \neg \neg X(TU\neg P), \neg X(TU\neg P), \neg X((TU\neg P), \neg X(TU\neg P)) \} \\
K_9 &= \{ T, P, \neg \neg \neg X(TU\neg P), \neg X(TU\neg P), \neg X((TU\neg P), \neg X(TU\neg P)) \} \\
K_{10} &= \{ T, P, \neg \neg \neg X(TU\neg P), \neg X(TU\neg P), \neg X((TU\neg P), \neg X(TU\neg P)) \}
\end{align*}
\]

Thus we obtain the atoms \( (b, K_7), (b, K_8), (b, K_{10}), (c, K_7), (c, K_8), (c, K_{10}) \), and \( (c, K_{10}) \).

Now we have the following “compatibility relation”:

\[
\begin{array}{cccccccccc}
K_1 & K_2 & K_3 & K_4 & K_5 & K_6 & K_7 & K_8 & K_9 & K_{10} \\
\hline
K_1 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_2 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_3 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_4 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_5 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_6 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_7 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_8 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_9 & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
K_{10} & \times & \times & \times & \times & \times & \times & \times & \times & \times \\
\end{array}
\]

The graph \( G \):

\[
\begin{align*}
& (a, K_1) \quad (b, K_7) \\
& (a, K_2) \quad (a, K_3) \quad (c, K_7) \\
& (b, K_3) \quad (c, K_3) \\
& (b, K_{10}) \quad (c, K_{10}) \\
& (a, K_3) \quad (b, K_3) \quad (c, K_3) \quad (a, K_3) \\
& (a, K_3) \quad (a, K_3) \quad (a, K_3) \\
& (a, K_3) \quad (a, K_3) \\
& (a, K_3) \quad (a, K_3) \\
\end{align*}
\]

The non-trivial SCCs of \( G \):

\[
\begin{align*}
C_1 &= \{ (a, K_1), (a, K_3) \} \\
C_2 &= \{ (a, K_2), (a, K_3) \} \\
C_3 &= \{ (b, K_2), (c, K_7) \} \\
C_4 &= \{ (b, K_3), (c, K_3) \} \\
C_5 &= \{ (b, K_3), (c, K_5) \}
\end{align*}
\]

Out of these, \( C_2 \) and \( C_5 \) are self-fulfilling. Now check whether \( C_2 \) or \( C_5 \) is reachable from some atom \( (a, K) \), where \( \neg (TU\neg (TU\neg P)) \in K \).

Since the formula \( \neg (TU\neg (TU\neg P)) \) is in \( K_6 \) and \( C_2 \) is reachable from
Since \((a, K_6) \in C_2\), it follows that \(\mathcal{M}, a \models E \neg FG P\) holds. Thus \(\mathcal{M}, a \not\models AFG P\) does not hold.

1. We start from the negation of the given formula,

\[
\neg \left( (Q \lor (P \land AXA (P \lor Q))) \rightarrow A (P \lor Q) \right).
\]

and translate it to positive normal form:

\[
\begin{align*}
&\left( Q \lor (P \land AXA (P \lor Q)) \right) \land \neg A (P \lor Q) \\
&\left( Q \lor (P \land AXA (P \lor Q)) \right) \land E (\neg P \lor B \lor Q)
\end{align*}
\]

Now construct the CTL tableau. We start with the OR-node

\[D_0 = \left\{ \left( Q \lor (P \land AXA (P \lor Q)) \right) \land E (\neg P \lor B \lor Q) \right\},\]

the AND-successors of which are

\[C_0 = \left\{ \left( Q \lor (P \land AXA (P \lor Q)) \right) \land E (\neg P \lor B \lor Q), \right.\]

\[\begin{align*}
Q \lor (P \land AXA (P \lor Q)) &\text{, } E (\neg P \lor B \lor Q), \\
Q, \neg Q, &\neg P \lor E (\neg P \lor B \lor Q), \neg P
\end{align*}\]

\[C_1 = \left\{ \left( Q \lor (P \land AXA (P \lor Q)) \right) \land E (\neg P \lor B \lor Q), \right.\]

\[\begin{align*}
Q \lor (P \land AXA (P \lor Q)) &\text{, } E (\neg P \lor B \lor Q), \\
Q, \neg Q, &\neg P \lor E (\neg P \lor B \lor Q), \neg P \lor E (\neg P \lor B \lor Q)
\end{align*}\]

\[C_2 = \left\{ \left( Q \lor (P \land AXA (P \lor Q)) \right) \land E (\neg P \lor B \lor Q), \right.\]

\[\begin{align*}
Q \lor (P \land AXA (P \lor Q)) &\text{, } E (\neg P \lor B \lor Q), \\
P \land AXA (P \lor Q), &P, AXA (P \lor Q), \neg Q, \neg P \lor E (\neg P \lor B \lor Q), \neg P
\end{align*}\]

\[C_3 = \left\{ \left( Q \lor (P \land AXA (P \lor Q)) \right) \land E (\neg P \lor B \lor Q), \right.\]

\[\begin{align*}
Q \lor (P \land AXA (P \lor Q)) &\text{, } E (\neg P \lor B \lor Q), \\
P \land AXA (P \lor Q), &P, AXA (P \lor Q), \neg Q, \neg P \lor E (\neg P \lor B \lor Q), \neg P \lor E (\neg P \lor B \lor Q)
\end{align*}\]
The nodes $C_0$, $C_1$, and $C_2$ can be pruned, since they are contradictory (contain the formulas $\phi$ and $\neg \phi$ for some $\phi$). (In other words, we can in fact prune contradictory AND-nodes already when constructing AND-nodes without affecting the end result! Left as an exercise.)

Since the node $C_3$ contains the formulas $\mathbf{AXA}(PUQ)$ and $\mathbf{EXE}(\neg PBQ)$, for $C_3$ we have the OR-successor $D_1 = \{ \mathbf{A}(PUQ), \mathbf{E}(\neg PBQ) \}$.

The AND-successors of $D_1$:  

- $C_4 = D_1 \cup \{ Q, \neg Q, \neg P \lor \mathbf{EXE}(\neg PBQ), \neg P \}$
- $C_5 = D_1 \cup \{ Q, \neg Q, \neg P \lor \mathbf{EXE}(\neg PBQ), \mathbf{EXE}(\neg PBQ) \}$
- $C_6 = D_1 \cup \{ P \land \mathbf{AXA}(PUQ), P, \mathbf{AXA}(PUQ), \neg Q, \neg P \lor \mathbf{EXE}(\neg PBQ), \neg P \}$

The nodes $C_4$, $C_5$, and $C_6$ are contradictory and hence pruned. We are left with $C_7$ for which we obtain the OR-successor $\{ \mathbf{A}(PUQ), \mathbf{E}(\neg PBQ) \} = D_1$.

The AND-node $C_7$ is pruned since it contains the eventuality formula $\mathbf{A}(PUQ)$ which is not satisfiable. (Since $C_7$ does not contain $Q$, we have that $D_1$ must be in the required acyclic subgraph that would imply that $\mathbf{A}(PUQ)$ is satisfiable. However, since $C_7$ is the only successor of $D_1$, there is no such acyclic subgraph.)

After removing $C_2$, we can step-by-step remove the nodes $D_1, C_3$, and $D_0$. Since the resulting tableau does not contain an AND-node which contains the formula 

$$\left( Q \lor (P \land \mathbf{AXA}(PUQ)) \right) \land \mathbf{E}(\neg PBQ),$$

it follows that the formula is unsatisfiable. Hence the negation of this formula (the original formula given in the assignment) is valid.

2. Determine the positive normal form:

$$\begin{align*}
\mathbf{GF}P &\rightarrow \mathbf{GF} \neg P \\
\neg \mathbf{GF}P &\lor \mathbf{GF} \neg P \\
\mathbf{FG} \neg P &\lor \mathbf{GF} \neg P
\end{align*}$$

Then, replace the LTL connectives $F$ and $G$ with the CTL connective $\mathbf{AF}$ and $\mathbf{AG}$, respectively. We obtain the formula

$$\mathbf{AF}\mathbf{G} \neg P \lor \mathbf{AGAF} \neg P.$$ 

This CTL formula is satisfiable if and only if the original LTL formula is satisfiable. We can thus apply the CTL tableau method. The root of the tableau is the OR-node $D_0 = \{ \mathbf{A}(\mathbf{G}P \lor \neg \mathbf{A}(\mathbf{G}P \lor \neg \mathbf{A}(\mathbf{G}P \lor \neg \mathbf{A}(\mathbf{G}P \lor \neg \mathbf{A}))) \}$.

The AND-successors of $D_0$:  

- $C_0 = \{ \mathbf{AFAG} \neg P \lor \mathbf{AGAF} \neg P, \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$
- $C_1 = \{ \mathbf{AFAG} \neg P \lor \mathbf{AGAF} \neg P, \mathbf{AFAG} \neg P, \mathbf{AXAFAG} \neg P \}$
- $C_2 = \{ \mathbf{AFAG} \neg P \lor \mathbf{AGAF} \neg P, \mathbf{AGAF} \neg P, \mathbf{AF} \neg P, \mathbf{AXAGAF} \neg P, \neg P \}$

The OR-successor of $C_0$:  

- $D_0 = \{ \mathbf{AFAG} \neg P \lor \mathbf{AGAF} \neg P \}$

The AND-successors of $C_0$:  

- $C_1 = \{ \mathbf{AFAG} \neg P \lor \mathbf{AGAF} \neg P, \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$

The OR-successor of $C_1$:  

- $D_1 = \{ \mathbf{AFAG} \neg P \}$

The AND-successors of $C_1$:  

- $C_2 = \{ \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$

The AND-successors of $C_2$:  

- $C_3 = \{ \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$

The AND-successors of $C_3$:  

- $C_4 = \{ \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$

The AND-successors of $C_4$:  

- $C_5 = \{ \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$

The AND-successors of $C_5$:  

- $C_6 = \{ \mathbf{AFAG} \neg P, \mathbf{AG} \neg P, \neg P, \mathbf{AXAG} \neg P \}$

The AND-successors of $C_6$:  

- $C_7 = \{ \mathbf{AFAG} \neg P, \mathbf{AF} \neg P, \mathbf{AXAGAF} \neg P, \neg P \}$

The AND-successors of $C_7$:  

- $C_8 = \{ \mathbf{AFAG} \neg P, \mathbf{AF} \neg P, \mathbf{AXAGAF} \neg P, \mathbf{AXAF} \neg P \}$

The AND-successors of $C_8$:  

- $C_9 = \{ \mathbf{AFAG} \neg P, \mathbf{AF} \neg P, \mathbf{AXAGAF} \neg P, \mathbf{AXAF} \neg P \}$

The AND-successors of $C_9$:  

- $C_{10} = \{ \mathbf{AFAG} \neg P, \mathbf{AF} \neg P, \mathbf{AXAGAF} \neg P, \mathbf{AXAF} \neg P \}$
The OR-successor of $C_4$: $D_5 = \{AG\neg P\} = D_1$

The OR-successor of $C_5$: $D_6 = \{AG\neg P\} = D_1$

The OR-successor of $C_6$: $D_7 = \{AFAG\neg P\} = D_2$

The OR-successor of $C_7$: $D_8 = \{AGAF\neg P\} = D_3$

The OR-successor of $C_8$: $D_9 = \{AGAF\neg P, AF\neg P\} = D_4$

The initial tableau $T_0$:

Since the node $C_4$ is not contradictory and does not contain any eventuality formulas, it remains in the final tableau. Thus the OR-node $D_1$ will have a successor, and hence $D_1$ will remain as well. All successors of node $C_0$ will then also remain. Furthermore, node $C_0$ remains since it is not contradictory and all of its eventuality formulas (there is only one, $AFAG\neg P$) are satisfiable in the initial tableau.

It follows that the final tableau obtained from $T_0$ contains the AND-node $C_0$ which contains the formula $AFAG\neg P \lor AGAF\neg P$. Hence this CTL formula is satisfiable.

Using the nodes $C_0$ and $C_4$ we can now construct a model for the CTL formula:

$C_0 \rightarrow \rightarrow C_4 \leftarrow \leftarrow \neg P \neg P$

Since the CTL formula $AFAG\neg P \lor AGAF\neg P$ is satisfiable, the original LTL formula $FG\neg P \lor GF\neg P \equiv GF P \rightarrow GF\neg P$ is also satisfiable.