TABLEAU METHOD FOR MODAL LOGICS

1. Tableau method for modal logic $\mathbf{K}$
2. Soundness
3. Completeness
4. Other modal logics
5. Logical consequence

M. Fitting: *Basic Modal Logic*, 1.9 (pp. 396 – 403).

1. Tableau Method of Modal Logic $\mathbf{K}$

- **Motivation**: Hilbert-style proofs are very hard to discover.

  Applying the Modus Ponens rule: in order to derive Q, one needs to derive P and $P \to Q$ first. But what is an appropriate P?

  Example. Does $\{A \to B,\ A \to C\} \models A \to (B \wedge C)$ hold?

  1. $A \to B$
  2. $A \to C$

  ...

  n. $(A \to B) \to ((A \to C) \to (A \to (B \wedge C)))$
- **Subformula principle:** To derive a formula Q it is sufficient to consider only the subformulas of Q (or their negations)
- Proof systems obeying this principle are more suitable for automation:

  $\Longrightarrow$ Resolution, sequent calculi, tableau methods.

Naming Possible Worlds

- The idea is to name possible worlds using prefixes in such a way that it is possible to determine the accessibility relation between worlds by their prefixes.
- A prefix is a non-empty finite sequence of natural numbers. For example, $\langle 1 \rangle$ and $\langle 11, 1, 1, 1, 111, 2 \rangle$ are prefixes.
- A prefixed formula is an expression of the form σP where σ is a prefix and P is a formula.

  (The idea: the formula P is true in the world named by σ.)

  Examples of prefixed formulas: $\langle 1 \rangle (P \vee \neg P)$, $\langle 11, 1, 1, 111, 2 \rangle \Box\Box\Diamond P$
- Notation: σn is a prefix which is obtained from the prefix σ by appending to σ the number n. For example, if $\sigma = \langle 1 \rangle$, then $\sigma 11 = \langle 1, 11 \rangle$.
- A prefix of the form σn is $\mathbf{K}$-accessible from the prefix σ.

  For example, $\langle 1, 11, 11 \rangle$ $\mathbf{K}$-accessible from $\langle 1, 11 \rangle$.

Tableau Rules

The tableau method for modal logics consists of the tableau rules for the classical propositional logic and modal rules.

- Tableau rules for classical propositional logic and prefixed formulas (see the α and β rules in the course refresher):

$$\frac{\sigma \neg\neg P}{\sigma P} \qquad \frac{\sigma \alpha}{\begin{array}{c}\sigma \alpha_1 \\ \sigma \alpha_2\end{array}} \qquad \frac{\sigma \beta}{\sigma \beta_1 \mid \sigma \beta_2}$$

Remark. The prefix σ does not change.

Example

Applying the tableau rules for propositional logic for the prefixed formula $\langle 3,2,1\rangle\neg((\neg P \to Q) \to P)$:

1. $\langle 3,2,1\rangle\neg((\neg P \to Q) \to P)$
2. $\langle 3,2,1\rangle(\neg P \to Q)$ (1)
3. $\langle 3,2,1\rangle\neg P$ (1)

4. $\langle 3,2,1\rangle\neg\neg P$ (2)	5. $\langle 3,2,1\rangle Q$ (2)
6. $\langle 3,2,1\rangle P$ (4)	

Occurrences of Prefixes

Definition. A prefix on a branch of a tableau is

(i) available if it appears on the branch and

(ii) unrestricted if it is not an initial segment (proper or improper) of any prefix occurring on the branch.

Example. The prefix $\langle 1,1\rangle$ is an initial segment of $\langle 1,1,12,3\rangle$ and $\langle 1,1\rangle$.

Modal Rules

For modal logic $\mathbf{K}$ the following modal rules are used:

$\Box$ **Rule:** $$\frac{\sigma\Box P}{\sigma n P}$$

for any available prefix σn.

$\neg\Box$ **Rule:** $$\frac{\sigma\neg\Box P}{\sigma n\neg P}$$

for an unrestricted prefix σn.

Example

1. $\langle 1\rangle\neg(\Box P \to \Box\Box P)$
2. $\langle 1\rangle\Box P$ (1)
3. $\langle 1\rangle\neg\Box\Box P$ (1)
4. $\langle 1,1\rangle\neg\Box P$ (3)
5. $\langle 1,1\rangle P$ (2)
6. $\langle 1,1,1\rangle\neg P$ (4)
7. $\langle 1,2\rangle\neg\Box P$ (3)
8. $\langle 1,2,3\rangle\neg P$ (7)
9. $\langle 1,2\rangle P$ (2)

$\vdots$

Tableau Proofs

Definition.

- A branch of a tableau is closed if it contains
  1. prefixed formulas σP and $\sigma\neg P$ for some formula P and prefix σ; or
  2. a prefixed formula $\sigma\bot$ or a prefixed formula $\sigma\neg\top$ for some prefix σ.
- A tableau is closed if every branch in it is closed.
- A proof of a formula P is a tableau that has been constructed with the prefixed formula $\langle 1\rangle\neg P$ as the root and that is closed.

Example

A tableau proof for the formula $\Box(P\wedge Q)\rightarrow(\Box P\wedge\Box Q)$:

1. $\langle 1\rangle\neg(\Box(P\wedge Q)\rightarrow(\Box P\wedge\Box Q))$
2. $\langle 1\rangle\Box(P\wedge Q)$ (1)
3. $\langle 1\rangle\neg(\Box P\wedge\Box Q)$ (1)

4.	$\langle 1\rangle\neg\Box P$	(3)	5.	$\langle 1\rangle\neg\Box Q$	(3)
6.	$\langle 1,2\rangle\neg P$	(4)	10.	$\langle 1,3\rangle\neg Q$	(5)
7.	$\langle 1,2\rangle P\wedge Q$	(2)	11.	$\langle 1,3\rangle P\wedge Q$	(2)
8.	$\langle 1,2\rangle P$	(7)	12.	$\langle 1,3\rangle P$	(11)
9.	$\langle 1,2\rangle Q$	(7)	13.	$\langle 1,3\rangle Q$	(11)
	$\times$			$\times$	

Example

A tableau proof for the formula $\top\leftrightarrow\Box\top$:

1. $\langle 1\rangle\neg(\top\leftrightarrow\Box\top)$

2.	$\langle 1\rangle\neg\top$	(1)	4.	$\langle 1\rangle\top$	(1)
3.	$\langle 1\rangle\Box\top$	(1)	5.	$\langle 1\rangle\neg\Box\top$	(1)
	$\times$		6.	$\langle 1,2\rangle\neg\top$	(5)
				$\times$	

2. Soundness

Theorem. If there is a tableau proof for a formula P, then P is $\mathbf{K}$-valid.

Proof. (using the following steps)

- We define the concept of $\mathbf{K}$-satisfiability for a tableau: a tableau is $\mathbf{K}$-satisfiable if it has a branch that corresponds to a model in a particular way (to be defined below).
- Then we show that
  - (i) a $\mathbf{K}$-satisfiable tableau cannot be closed.
  - (ii) If a formula P is not $\mathbf{K}$-valid, then the tableau consisting only of a root node $\langle 1\rangle\neg P$ is $\mathbf{K}$-satisfiable.
  - (iii) Every tableau rule preserve $\mathbf{K}$-satisfiability: is a tableau is $\mathbf{K}$-satisfiable before applying a rule, then it is $\mathbf{K}$-satisfiable after applying the rule.

Hence, if P is not $\mathbf{K}$-valid, the tableau for P remains open. ■

K-Satisfiability of a Tableau

Definition.

- A tableau is **K**-satisfiable if one of its branches is **K**-satisfiable.
- A branch of a tableau is **K**-satisfiable if the set of prefixed formulas occurring on the branch is **K**-satisfiable.
- A set of prefixed formulas Σ is **K**-satisfiable if there is a model $\mathcal{M} = \langle S, R, v\rangle$ and a mapping $\mathcal{N}$ from the prefixes appearing in Σ to the set S such that
  1. If prefixes σ and τ occur in the set Σ and τ is **K**-accessible from σ, then $\mathcal{N}(\sigma) R \mathcal{N}(\tau)$.
  2. If $\sigma P \in \Sigma$, then $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- P$.

A More Detailed Proof

Based on the definition above we can show the following:

(i) A **K**-satisfiable tableau cannot be closed.

Assume that a **K**-satisfiable tableau is closed. By **K**-satisfiability the tableau has a **K**-satisfiable branch. Also this branch is closed, i.e., contains prefixed formulas σQ and $\sigma\neg Q$ (or $\sigma\bot$ or $\sigma\neg\top$). But this leads to a contradiction because for the branch there is a model $\mathcal{M}$ and a mapping $\mathcal{N}$ such that $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- Q$ and $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- \neg Q$ (or $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- \bot$ or $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- \neg\top$) which is impossible. Hence (i) holds.

A More Detailed Proof—cont'd

(ii) If a formula P is not **K**-valid, then the tableau containing only the root node $\langle 1\rangle\neg P$ is **K**-satisfiable.

This holds because if the formula P is not **K**-valid, it has a counter-model $\mathcal{M} = \langle S, R, v\rangle$ such that for some world $s \in S$, $\mathcal{M}, s \mid\mid\!\!- \neg P$. Hence, the tableau containing only the root node and the branch $\{\langle 1\rangle\neg P\}$ is **K**-satisfiable using the mapping $\mathcal{N}(\langle 1\rangle) = s$.

(iii) Let a tableau Γ be **K**-satisfiable. Then it has a branch whose set of prefixed formulas Σ is **K**-satisfiable for some model $\mathcal{M} = \langle S, R, v\rangle$ and mapping $\mathcal{N}$.

We show that if a tableau rule is applied to Γ, then the resulting tableau Γ' also **K**-satisfiable.

Tableau Rules Preserve K-Satisfiability

Consider a tableau rule that is applied to a **K**-satisfiable branch (and let Σ be the set of prefixed formulas on the branch):

- $$\frac{\sigma\beta}{\sigma\beta_1 \mid \sigma\beta_2}$$

  Then the tableau Γ' has two branches with the sets of prefixed formulas $\Sigma_1 = \Sigma \cup \{\sigma\beta_1\}$ and $\Sigma_2 = \Sigma \cup \{\sigma\beta_2\}$.

  Since $\sigma\beta \in \Sigma$, $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- \beta$,

  and, thus, $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- \beta_1$ or $\mathcal{M}, \mathcal{N}(\sigma) \mid\mid\!\!- \beta_2$.

  $\Longrightarrow$ Γ' is **K**-satisfiable.

- $$\frac{\sigma\alpha}{\begin{array}{l}\sigma\alpha_1\\ \sigma\alpha_2\end{array}}$$
  can be shown in a similar way as the resulting tableau contains the set of prefixed formulas $\Sigma_1 = \Sigma \cup \{\sigma\alpha_1, \sigma\alpha_2\}$.

- $$\frac{\sigma\Box P}{\sigma n P}$$ for some available prefix σn.

  Then the resulting tableau Γ' has a branch whose set of prefixed formulas is $\Sigma_1 = \Sigma \cup \{\sigma n P\}$.

  As $\sigma\Box P \in \Sigma$, then $\mathcal{M}, \mathcal{N}(\sigma) \Vdash \Box P$, and, hence, for all t such that $\mathcal{N}(\sigma) R t$ $\mathcal{M}, t \Vdash P$ holds.

  Since σn occurs on the branch and is $\mathbf{K}$-accessible from σ, then $\mathcal{N}(\sigma) R \mathcal{N}(\sigma n)$ holds. Hence, $\mathcal{M}, \mathcal{N}(\sigma n) \Vdash P$ holds.

  $\Longrightarrow$ Γ' on $\mathbf{K}$-satisfiable.

- $$\frac{\sigma\neg\Box P}{\sigma n \neg P}$$ for some unrestricted prefix σn.

  Then the resulting tableau Γ' has a branch whose set of prefixed formulas $\Sigma_1 = \Sigma \cup \{\sigma n \neg P\}$.

  Since $\sigma\neg\Box P \in \Sigma$, $\mathcal{M}, \mathcal{N}(\sigma) \Vdash \neg\Box P$, and, thus, there is a world t such that $\mathcal{N}(\sigma) R t$ and $\mathcal{M}, t \Vdash \neg P$.

  Because σn is not an initial segment of any prefix in Σ, the mapping $\mathcal{N}$ can be extended: $\mathcal{N}(\sigma n) = t$. Then $\mathcal{M}, \mathcal{N}(\sigma n) \Vdash \neg P$ holds.

  $\Longrightarrow$ Γ' on $\mathbf{K}$-satisfiable.

3. Completeness

- How to guarantee that every valid formula has a tableau proof when the branches of a tableau can be infinite?
- When have the rules been applied fairly/enough times?
- For constructing tableaux a systematic method is needed where all the rules have been applied sufficiently, i.e., for every open branch θ it holds that
  (i) If $\sigma\neg\neg P \in \theta$, then $\sigma P \in \theta$.
  (ii) If $\sigma\beta \in \theta$, then $\sigma\beta_1 \in \theta$ or $\sigma\beta_2 \in \theta$.
  (iii) If $\sigma\alpha \in \theta$, then $\sigma\alpha_1 \in \theta$ and $\sigma\alpha_2 \in \theta$.
  (iv) If $\sigma\neg\Box Q \in \theta$, then $\sigma n \neg Q \in \theta$, for some n.
  (v) If $\sigma\Box Q \in \theta$, $\sigma n Q \in \theta$ for all σn available on θ.

Observations

- The systematic method guarantees completeness but can allow infinite tableau if the formula considered is not valid.
- A **decision method** guarantees that the construction of the tableau terminates after a finite number of step regardless whether the considered formula is valid or not.
- We will considered the question about decision methods later.

Systematic K-Tableau for a Formula P:

1. Take as the root of the tableau $\langle 1\rangle\neg P$.
2. Until the tableau is closed or all formulas have been marked used do:

2.1 Choose the top unused node σQ in the tableau.

2.2 If Q is not a literal, then for every open branch θ containing σQ do:

- If σQ is of the form $\sigma\neg\neg Q'$, extend θ by the node $\sigma Q'$.
- If σQ is of the form $\sigma\alpha$, extend θ by nodes $\sigma\alpha_1$ and $\sigma\alpha_2$.
- If σQ is of the form $\sigma\beta$, extend θ to two branches one containing $\sigma\beta_1$ and the other $\sigma\beta_2$.
- If σQ is of the form $\sigma\neg\Box P$, extend θ by $\sigma n\neg P$ for some prefix σn unrestricted in θ.
- If σQ is of the form $\sigma\Box P$, extend θ for all prefixes σn available on θ with $\sigma n P$ and $\boxed{\text{then with the node } \sigma\Box P}$.

2.3 Mark σQ used.

Theorem. (**Completeness**) If a formula P is $\mathbf{K}$-valid, then the systematic $\mathbf{K}$-tableau for P will be closed.

Proof. We show that if the systematic $\mathbf{K}$-tableau for P has an open branch, then P is not $\mathbf{K}$-valid. Let θ be such a branch and $\mathcal{M} = \langle S, R, v\rangle$ a counter-model based on it:

1. S is the set of prefixes occurring in θ.
2. $\sigma R\tau$ iff τ is $\mathbf{K}$-accessible from σ
3. $v(\sigma, Q) = \text{true}$ iff σQ occurs on the branch θ for every atomic proposition Q.

To prove the theorem it is enough to show the following lemma:

$$\text{if } \sigma Q \in \theta, \text{ then } \mathcal{M}, \sigma \Vdash Q.$$

This implies the theorem because $\langle 1\rangle\neg P$ occurs on every branch and, thus, $\mathcal{M}, \langle 1\rangle \Vdash \neg P$. This implies that P is not $\mathbf{K}$-valid. ∎

Induction Proof of the Lemma

We show for all prefixed formulas σQ that

$$\text{if } \sigma Q \in \theta, \text{ then } \mathcal{M}, \sigma \Vdash Q.$$

Proof. This is done by induction on the length of the formula Q.

- (Q is an atomic proposition) If $\sigma Q \in \theta$, then $v(\sigma, Q) = \text{true}$ and $\mathcal{M}, \sigma \Vdash Q$.
- (Q is the negation of an atomic proposition) If $\sigma\neg Q' \in \theta$ and Q' is an atomic proposition, then $\sigma Q' \notin \theta$ and, hence, $\mathcal{M}, \sigma \nVdash Q'$ and $\mathcal{M}, \sigma \Vdash \neg Q'$.

Induction hypothesis [IH]:

$$\text{If } Q \text{ shorter that } j \text{ and } \sigma Q \in \theta, \text{ then } \mathcal{M}, \sigma \Vdash Q.$$

Let the length of Q be j. Then Q is one of the following forms

- ($\neg\neg Q$) If $\sigma\neg\neg Q \in \theta$, then $\sigma Q \in \theta$.
  By [IH] $\mathcal{M}, \sigma \Vdash Q$. Hence, $\mathcal{M}, \sigma \Vdash \neg\neg Q$.
- (β-formula) If $\sigma\beta \in \theta$, then $\sigma\beta_1 \in \theta$ or $\sigma\beta_2 \in \theta$.
  By [IH] $\mathcal{M}, \sigma \Vdash \beta_1$ or $\mathcal{M}, \sigma \Vdash \beta_2$. Thus, $\mathcal{M}, \sigma \Vdash \beta$.
- (α-formula) If $\sigma\alpha \in \theta$, then $\sigma\alpha_1 \in \theta$ and $\sigma\alpha_2 \in \theta$.
  By [IH] $\mathcal{M}, \sigma \Vdash \alpha_1$ and $\mathcal{M}, \sigma \Vdash \alpha_2$. Hence, $\mathcal{M}, \sigma \Vdash \alpha$.
- ($\neg\Box Q$) If $\sigma\neg\Box Q \in \theta$, $\sigma n\neg Q \in \theta$ for some n.
  By [IH] $\mathcal{M}, \sigma n \Vdash \neg Q$. So $\mathcal{M}, \sigma \nVdash \Box Q$ as $\sigma R\sigma n$.
- ($\Box Q$) If $\sigma\Box Q \in \theta$, $\sigma n Q \in \theta$ for all σn available on θ. By [IH] $\mathcal{M}, \sigma n \Vdash Q$. Hence, $\mathcal{M}, \sigma \Vdash \Box Q$.

So for every open branch θ, prefix σ and a formula Q:
if $\sigma Q \in \theta$, then $\mathcal{M}, \sigma \Vdash Q$. ∎

Example. We study whether $\Box P \rightarrow \Box\Box P$ is $\mathbf{K}$-valid by constructing a systematic $\mathbf{K}$-tableau:

1.	$\langle 1\rangle\neg(\Box P \rightarrow \Box\Box P)$	
2.	$\langle 1\rangle\Box P$	(1)
3.	$\langle 1\rangle\neg\Box\Box P$	(1)
4.	$\langle 1\rangle\Box P$	(2)
5.	$\langle 1,2\rangle\neg\Box P$	(3)
6.	$\langle 1,2\rangle P$	(4)
7.	$\langle 1\rangle\Box P$	(4)
8.	$\langle 1,2,3\rangle\neg P$	(5)
9.	$\langle 1,2\rangle P$	(7)
10.	$\langle 1\rangle\Box P$	(7)
	...	

As the tableau cannot be closed, we can construct from the open branch a *counter-model* $\mathcal{M} = \langle S, R, v\rangle$ where
$S = \{\langle 1\rangle, \langle 1,2\rangle, \langle 1,2,3\rangle\}$
$R = \{(\langle 1\rangle, \langle 1,2\rangle), (\langle 1,2\rangle, \langle 1,2,3\rangle)\}$
and $v(\sigma, P) = \text{true}$ iff $\sigma = \langle 1,2\rangle$.
Now $\mathcal{M}, \langle 1\rangle \not\Vdash \Box P \rightarrow \Box\Box P$.

4. Other Modal Logics

- We extend the tableau method for $\mathbf{K}$ to handle also other modal logics.

  Definition. A prefix τ is a simple extension of a prefix σ, if τ is of the form σn for some natural number n.

- The tableau method of a modal logic $\mathbf{L}$:

  $\neg\Box$ rule:

  $$\frac{\sigma\neg\Box P}{\tau\neg P} \qquad \frac{\sigma\Diamond P}{\tau P}$$

  where τ is a simple extension of the prefix σ unrestricted on the branch.

Tableau Method of a Modal Logic L—cont'd

- **$\Box$ rule**:

  $$\frac{\sigma\Box P}{\tau P} \qquad \frac{\sigma\neg\Diamond P}{\tau\neg P}$$

  where τ $\mathbf{L}$-accessible from the prefix σ and

  1. for logics $\mathbf{K}, \mathbf{KB}, \mathbf{K4}$ (Obs. non-serial)
     τ is available on the branch;
  2. for logics $\mathbf{D}, \mathbf{T}, \mathbf{DB}, \mathbf{B}, \mathbf{D4}, \mathbf{S4}, \mathbf{S5}$ (Obs. serial)
     (a) τ is available on the branch or
     (b) τ is a simple extension of σ unrestricted on the branch.

Accessibility between Prefixes

For defining $\mathbf{L}$-accessibility between prefixes we need a couple of additional concepts.

Definition. An accessibility relation between prefixes is

1. general if σn is accessible from σ for all n;
2. reverse if σ is accessible from σn for all n;
3. reflexive if σ is accessible from itself.
4. transitive if τ is accessible from σ whenever σ is a proper initial segment of τ.
5. universal if a prefix is accessible from any prefix.

L-Accessibility of Prefixes–cont'd

L-Accessibility for some logics **L**:

Logic **L**	**L**-accessibility
K, **D**	general
T	general, reflexive
KB, **DB**	general, reverse
B	general, reflexive, reverse
K4, **D4**	general, transitive
S4	general, reflexive, transitive
S5	universal

Example: D-validity

We search for a **D**-tableau proof for the formula $\Box P \to \neg\Box\neg P$:

1. $\langle 1\rangle\neg(\Box P \to \neg\Box\neg P)$
2. $\langle 1\rangle\Box P$ (1)
3. $\langle 1\rangle\neg\neg\Box\neg P$ (1)
4. $\langle 1\rangle\Box\neg P$ (3)
5. $\langle 1,2\rangle\neg P$ (4) Observe: 2. (b)
6. $\langle 1,2\rangle P$ (2)

$\times$

(**D**-accessibility: general)

Example: T-validity

We construct a **T**-tableau proof for the formula$\Box P \to P$:

1. $\langle 1\rangle\neg(\Box P \to P)$
2. $\langle 1\rangle\Box P$ (1)
3. $\langle 1\rangle\neg P$ (1)
4. $\langle 1\rangle P$ (2) Obs. reflexivity

$\times$

(**T**-accessibility: general and reflexive)

Example: K4-validity

We search for a **K4** tableau proof for the formula $\Box P \to \Box\Box P$:

1. $\langle 1\rangle\neg(\Box P \to \Box\Box P)$
2. $\langle 1\rangle\Box P$ (1)
3. $\langle 1\rangle\neg\Box\Box P$ (1)
4. $\langle 1,2\rangle\neg\Box P$ (3)
5. $\langle 1,2,3\rangle\neg P$ (4)
6. $\langle 1,2,3\rangle P$ (2)Obs. transitivity

$\times$

(**K4**-accessibility: general and transitive)

Example: KB-validity

We search for a **KB**-tableau proof for $P \rightarrow \Box\Diamond P$:

1. $\langle 1\rangle \neg(P \rightarrow \Box\Diamond P)$
2. $\langle 1\rangle P$ (1)
3. $\langle 1\rangle \neg\Box\Diamond P$ (1)
4. $\langle 1,2\rangle \neg\Diamond P$ (3)
5. $\langle 1\rangle \neg P$ (4) Obs. reverse

$\times$

(**KB**-accessibility: general and reverse)

A Systematic L-Tableau for a Formula P:

Change the systematic **K**-tableau point 2.2: If Q is not a literal, then for every open branch θ containing σQ:

- If σQ is of the form $\sigma\neg\Box P$ ($\sigma\Diamond P$), extend θ by $\sigma n\neg P$ ($\sigma n P$) for some prefix σn unrestricted in θ.
- If σQ is of the form $\sigma\Box P$ ($\sigma\neg\Diamond P$),
  (a) if **L** is one of **K**, **KB**, **K4**, **T**, **B**, **S4**, **S5**:
  for every prefix σ' available on θ that is **L**-accessible from σ, extend θ by $\sigma' P$ ($\sigma'\neg P$) and finally by $\sigma\Box P$ ($\sigma\neg\Diamond P$);
  (b) if **L** is one of **D**, **DB**, **D4**:
  for every prefix σ' available on θ that is **L**-accessible from σ, extend θ by $\sigma' P$ ($\sigma'\neg P$). If no such prefix σ' exists, extend θ by $\sigma n P$ ($\sigma n\neg P$) for some σn unrestricted in θ. In both cases add to the end of the branch θ $\sigma\Box P$ ($\sigma\neg\Diamond P$).

S5-tableaux

- In modal logic **S5** accessibility relation for models is an equivalence relation and it is enough to consider only universal frames (see, ML-4).
- As prefixes it is sufficient use natural numbers.

$\neg\Box$ rule: $\dfrac{n\neg\Box P}{k\neg P}$ $\dfrac{n\Diamond P}{kP}$

where k does not occur on the branch.

$\Box$ rule: $\dfrac{n\Box P}{kP}$ $\dfrac{n\neg\Diamond P}{k\neg P}$

for any k.

Example: S5-validity

We search of a **S5**-tableau proof for a $\neg\Box P \rightarrow \Box\neg\Box P$:

1. $1\neg(\neg\Box P \rightarrow \Box\neg\Box P)$
2. $1\neg\Box P$ (1)
3. $1\neg\Box\neg\Box P$ (1)
4. $2\neg P$ (2)
5. $3\neg\neg\Box P$ (3)
6. $3\Box P$ (5)
7. $2P$ (6)

$\times$

Systematic S5-tableau for a Formula P:

1. Take as the root of the tableau $1\neg P$.
2. Until the tableau is closed or all nodes have been marked used do:

2.1 Choose the topmost unused node nQ.

2.2 If Q is not a literal, then for every open branch θ including nQ:

- If σQ is of the form $\sigma\neg\neg Q'$, extend θ by the node $\sigma Q'$.
- If nQ is of the form $n\alpha$, extend θ by $n\alpha_1$ and $n\alpha_2$.
- If nQ is of the form $n\beta$, extend θ to two branches one containing $n\beta_1$ and the other $n\beta_2$.
- If nQ is of the form $n\neg\Box P$, extend θ by $k\neg P$ for some k unrestricted in θ and after this by kX for every $j\Box X$ on the branch.
- If nQ is of the form $n\Box P$ extend θ by adding for all k available on θ kP.

2.3 Mark nQ used.

KD45-Tableaux

- In modal logic **KD45** it is sufficient to consider models of the form (see. ML-4):

$$M = \langle \{s_0\} \cup S, \{\langle s,t\rangle \mid s \in \{s_0\} \cup S, t \in S\}, v\rangle.$$

- It is enough to use natural numbers as prefixes.

$\neg\Box$ rule: $\dfrac{n\neg\Box P}{k\neg P}$ $\qquad$ $\dfrac{n\Diamond P}{kP}$

where k does not occur on the branch.

$\Box$ rule: $\dfrac{n\Box P}{kP}$ $\qquad$ $\dfrac{n\neg\Diamond P}{k\neg P}$

for any $k \neq 1$.

Example: KD45-Validity

Compare the two **KD45**-tableaux:

$(\Box P \to P)$:

1.	$1\neg(\Box P \to P)$	
2.	$1\Box P$	(1)
3.	$1\neg P$	(1)

$\Box(\Box P \to P)$:

1.	$1\neg\Box(\Box P \to P)$	
2.	$2\neg(\Box P \to P)$	(1)
3.	$2\Box P$	(2)
4.	$2\neg P$	(2)
5.	$2P$	(3)

$\times$

5. Logical Consequence

- The task is to determine whether $\Sigma \models_{\mathbf{L}} \Upsilon \Longrightarrow P$ holds.
- In the tableau method we start in the usual way and take $\langle 1\rangle\neg P$ as the root node and then use the standard rules for the modal logic $\mathbf{L}$ and two new rules for the premises:

Global rule: A prefixed formula σQ can be added to any branch for any prefix σ available on the branch and for any global premise $Q \in \Sigma$.

Local rule: A prefixed formula $\langle 1\rangle Q$ can be added to any branch for any local premise $Q \in \Upsilon$.

Example: $\mathbf{K}$-Logical Consequence Relation

We show $\{P\} \models_{\mathbf{K}} \{\Box P \rightarrow Q\} \Longrightarrow Q$:

1. $\langle 1 \rangle \neg Q$
2. $\langle 1 \rangle \Box P \rightarrow Q$ (LP)

3. $\langle 1 \rangle \neg \Box P$ (2)	4. $\langle 1 \rangle Q$ (2)
5. $\langle 1,2 \rangle \neg P$ (2)	$\times$
6. $\langle 1,2 \rangle P$ (GP)	
$\times$	