**PROOF THEORY FOR MODAL LOGICS**

1. Hilbert-style proof theory
2. Soundness
3. Completeness
4. Generalization to local premises
5. Examples (T, S5 and KD45)

M. Fitting: *Basic Modal Logic*, 1.7 (pp. 387 – 391).

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**1. Hilbert-style Proof Theory**

For modal logic $K$:

**Classical axioms:** All (classical) tautologies.

**Modal axioms:** All formulas of the form

$$\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$$

**Modus Ponens Rule:**

$$\frac{P \rightarrow Q}{Q}$$

**Necessitation Rule (N-rule):**

$$\frac{P}{\Box P}$$

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**Derivation and Proof**

(We first consider the case without any local premises).

**Definition.** A $K$-derivation of a formula $P$ from a set of formulas $\Sigma$ is a finite sequence of formulas $\phi_1, \ldots, \phi_n$ such that $\phi_n = P$ and for all $i = 1, \ldots, n$

1. $\phi_i \in \Sigma$ or
2. $\phi_i$ is some axiom of $K$ or
3. $\phi_i$ is obtained by one of the rules Modus Ponens or Necessitation from earlier formulas in the sequence.

Notation: $\Sigma \vdash_K \theta \iff P$ (or $\Sigma \vdash_K P$)

A $K$-proof of a formula $P$ is a $K$-derivation of $P$ from the empty set of formulas.

Notation: $\emptyset \vdash_K \emptyset \iff P$ (or $\vdash_K P$)
A $K$-proof for $T \leftrightarrow \Box T$:

1. $T$ (Taut)
2. $\Box T \rightarrow (T \rightarrow \Box T)$ (Taut)
3. $\Box T$ (N, 1)
4. $T \rightarrow \Box T$ (MP, 2, 3)
5. $T \rightarrow (\Box T \rightarrow T)$ (Taut)
6. $\Box T \rightarrow T$ (MP, 1, 5)
7. $(\Box T \rightarrow T) \rightarrow ((T \rightarrow \Box T) \rightarrow (T \leftrightarrow \Box T))$ (Taut)
8. $(T \rightarrow \Box T) \rightarrow (T \leftrightarrow \Box T)$ (MP, 6, 7)
9. $T \leftrightarrow \Box T$ (MP, 4, 8)

Derived Rules (I)

Regularity Rule for $\Box$ (R-rule): $P \rightarrow Q$ $\Box P \rightarrow \Box Q$

1. $P \rightarrow Q$
2. $\Box (P \rightarrow Q)$ (N, 1)
3. $\Box (P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$ (K)
4. $\Box P \rightarrow \Box Q$ (MP, 2, 3)

Generalized R-rule: $P_1 \land \cdots \land P_n \rightarrow Q$ $\Box P_1 \land \cdots \land \Box P_n \rightarrow \Box Q$

2. Soundness

The soundness of a proof system for a logic:

If a formula is derivable in the proof system, it is also a logical consequence in the logic.

**Theorem.** If a formula $P$ has a $K$-derivation from a set of formulas $\Sigma$ ($\Sigma \vdash_K \theta \Rightarrow P$), then $\Sigma \models_K \theta \Rightarrow P$ (or in short: if $\Sigma \vdash_K P$, then $\Sigma \models_K P$).

**Proof.**

Let $\phi_1, \ldots, \phi_n (= P)$ be a $K$-derivation for a formula $P$.

We show by induction that for all $i = 1, \ldots, n$, $\Sigma \models_K \phi_i$ holds (i.e., $\phi_i$ is valid in every model where $\Sigma$ is valid).

So we prove by induction that for $i = 1, \ldots, n$, $C \models \phi_i$ holds where $C = \{M \mid M \models \Sigma\}$. 

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3. Completeness

Completeness of a proof systems for a logic:

If a formula is a logical consequence in the logic, then there is a derivation of it in the proof system.

Theorem. If \( \Sigma \vdash P \), then \( \Sigma \models_k P \).

The outline of the proof: Let \( \Sigma \not\models_k P \) hold.

We show that then also \( \Sigma \not\models_k P \) holds.

This is done by constructing a canonical model \( M \) where all formulas in \( \Sigma \) are valid and for every \( Q \) such that \( \Sigma \not\models_k Q \) holds there is a world \( s \) in \( M \) where \( M, s \models \neg Q \) holds.

The worlds of the model are maximally consistent sets of formulas that are constructed using Lindenbaum’s lemma from the set of premises \( \Sigma \).
We establish properties (i-iv) which imply the lemma.

(i) $A \subseteq \Delta$

(ii) for all $i = 0, 1, \ldots$, the set $\Delta_i$ is $\Sigma$-consistent.

Let $\Delta_{i-1}$ be $\Sigma$-consistent.

Assume that $\Delta_i$ is $\Sigma$-inconsistent. Then $\Delta_i = \Delta_{i-1} \cup \{\neg Q_{i-1}\}$ and $\Delta_{i-1} \cup \{Q_{i-1}\}$ are $\Sigma$-inconsistent.

Hence, there is a set $\{A_1^+, \ldots, A_n^+\} \subseteq \Delta_{i-1}$ such that

$\Sigma \vdash K \neg (T \wedge A_1^+ \wedge \cdots \wedge A_n^+ \wedge Q_{i-1})$

and a set $\{A_1^-, \ldots, A_n^-\} \subseteq \Delta_{i-1}$ such that

$\Sigma \vdash K \neg (T \wedge A_1^- \wedge \cdots \wedge A_n^- \wedge \neg Q_{i-1})$.

We can continue the derivations of the two formulas above:

1. $\neg (T \wedge A_1^+ \wedge \cdots \wedge A_n^+ \wedge Q_{i-1})$
2. $\neg (T \wedge A_1^- \wedge \cdots \wedge A_n^- \wedge \neg Q_{i-1})$
3. $Q_{i-1} \rightarrow \neg (T \wedge A_1^+ \wedge \cdots \wedge A_n^+)$ (Prop. 1)
4. $\neg Q_{i-1} \rightarrow \neg (T \wedge A_1^- \wedge \cdots \wedge A_n^-)$ (Prop. 2)
5. $\neg (T \wedge A_1^+ \wedge \cdots \wedge A_n^+) \lor$
   $\neg (T \wedge A_1^- \wedge \cdots \wedge A_n^-)$ (Prop. 3, 4)
6. $\neg (T \wedge A_1^+ \wedge \cdots \wedge A_n^+) \lor A_1^- \wedge \cdots \wedge A_n^-)$ (Prop. 5)

$\implies \Delta_{i-1}$ is $\Sigma$-inconsistent, a contradiction.
(ii) $\Delta$ is $\Sigma$-consistent.

Assume that $\Delta$ is $\Sigma$-inconsistent.

Hence, there is a set $\{A_1, \ldots, A_n\} \subseteq \Delta$ such that

$\Sigma \Gamma \not\vdash (T \land A_1 \land \cdots \land A_n).$

So there is $i \geq 0$ such that $\{A_1, \ldots, A_n\} \subseteq \Delta_i.$

$\implies \Delta_i$ is $\Sigma$-inconsistent, a contradiction.

(iv) $\Delta$ is maximally $\Sigma$-consistent.

Assume that $\Delta \cup \{Z\}$ $\Sigma$-consistent for some $Z \notin \Delta.$

As $Z = Q_i$ for some $i$, $\Delta \cup \{Q_i\}$ is $\Sigma$-consistent.

Because $\Delta \cup \{Q_i\} \subseteq \Delta \cup \{Q_i\}$ also $\Delta_i \cup \{Q_i\}$ is $\Sigma$-consistent [Lemma 1].

Thus, $Z \in \Delta_i + 1 \subseteq \Delta$, a contradiction.

(i-iv) $\implies \Delta$ is maximally $\Sigma$-consistent extension of the set $A.$

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**Important Lemmas—cont’d**

**Lemma 4.** For all maximally $\Sigma$-consistent sets $\Gamma$,

(i) $\Sigma \subseteq \Gamma$ and (ii) either $Z \in \Gamma$ or $\neg Z \in \Gamma$ for all formulas $Z$.

**Proof.** (i) Assume that $Z \in \Sigma \setminus \Gamma$. Then $\Gamma \cup \{Z\}$ is $\Sigma$-consistent and there is a set $\{A_1, \ldots, A_n\} \subseteq \Gamma$ such that $\Sigma \not\vdash \Gamma (T \land A_1 \land \cdots \land A_n \land Z).$

We can continue the derivation:

1. $\neg (T \land A_1 \land \cdots \land A_n \land Z)$
2. $Z \rightarrow \neg (T \land A_1 \land \cdots \land A_n)$ (Prop. 1)
3. $Z$ (GP)
4. $\neg (T \land A_1 \land \cdots \land A_n)$ (MP, 2, 3)

Hence, $\Sigma \not\vdash \Gamma (T \land A_1 \land \cdots \land A_n)$ holds.

$\implies \Gamma$ is $\Sigma$-inconsistent, a contradiction. Hence, $\Sigma \subseteq \Gamma$ holds.
• For atomic propositions $Q$ the claim holds by the definition of $M$.

• For a formula of the form $\neg Q$: $M,s \models \neg Q$ iff $M,s \not\models Q$ [H] $Q \notin s$ iff [Lemma 4 (ii)] $\neg Q \in s$.

• For a formula of the form $Q \rightarrow P$ the claim can be proved as above.

• For a formula of the form $\Box Q$:
  ($\Leftarrow$) Let $\Box Q \in s$ hold. If $s\tau t$, then $s^\# \subseteq t$, $Q \in t$ and $M,t \models Q$ [H]. Thus, $M,s \not\models \Box Q$.

  ($\Rightarrow$) Let $\Box Q \notin s$ hold. Then $\neg \Box Q \in s$ [Lemma 4 (ii)].

  Now $t_0 = s^\# \cup \{\neg Q\}$ is $\Sigma$-consistent [Lemma 2] and $t_0$ has a maximally $\Sigma$-consistent extension $t$ [Lemma 3 (Lindenbaum)].

  So $s\tau t$ because $s^\# \subseteq t$. As $\neg Q \in t_0 \subseteq t$, $Q \notin t$ holds [Lemma 4 (ii)].

  Hence, $M,t \not\models Q$ [H] and $M,s \not\models \Box Q$.

### Completeness Proof—Summary

• Because $\Sigma \subseteq s$ for all $s \in S$ [Lemma 4 (i)], the set $\Sigma$ is valid in the canonical model $M$ [Lemma 5].

• As the set $\{\neg P\}$ is $\Sigma$-consistent, the set has a maximal $\Sigma$-consistent extension $t \in S$ [Lemma 3] and $P \notin t$ [Lemma 4 (ii)].

  Thus, $M,t \not\models \Box P$ [Lemma 5].

• As $\Sigma$ is valid in $M = \langle S, R, v \rangle$ and there is a world $t \in S$ such that $M,t \not\models \Box P$ holds, also $\Sigma \not\models P$ holds.

  $\Rightarrow$ Hilbert-style proof theory for the modal logic $K$ is complete.

### 4. Generalization to Local Premises

**Definition.** $\Sigma \vdash_K \Upsilon \Rightarrow P$ means that there is a sequence of formulas ending with $P$ consisting of a global part, coming first, and a local part, coming last.

In the global part every formula is

• an axiom of $K$, belongs to the set $\Sigma$ or

• is obtained by one of the rules Modus Ponens or Necessitation from earlier formulas in the sequence.

In the local part every formula is

• an axiom of $K$, belongs to the set $\Upsilon$ or

• is obtained by the Modus Ponens rule from earlier formulas in the sequence.

### Example

We show that $\{P, P \rightarrow Q\} \vdash_K \{\Box Q \rightarrow R\} \Rightarrow R$ holds:

1. $P$ (GP)
2. $P \rightarrow Q$ (GP)
3. $Q$ (MP, 1, 2)
4. $\Box Q$ (N, 3)
5. $\Box Q \rightarrow R$ (LP)
6. $R$ (MP, 4, 5)

Note that the N-rule cannot be used in the local part.

For example, $\{P, P \rightarrow Q\} \vdash_K \{\Box Q \rightarrow R\} \Rightarrow \Box R$ does not hold.
Properties of Derivations

- Derivations are finite.
  \[ \Rightarrow \text{Compactness} \ (\vdash) : \]
  If \( \Sigma \vdash \Gamma \Rightarrow P \) holds, then there are finite sets \( \Sigma' \subseteq \Sigma \) and \( \Gamma' \subseteq \Gamma \) such that \( \Sigma' \vdash \Gamma' \Rightarrow P \) holds.
- MP- and N-rules are monotonic:
  \[ \Rightarrow \text{Monotonicity} \ (\vdash) : \]
  Let \( \Sigma_1 \subseteq \Sigma_2 \) and \( \Gamma_1 \subseteq \Gamma_2 \) hold. Then
  if \( \Sigma_1 \vdash \Gamma_1 \Rightarrow P \), then \( \Sigma_2 \vdash \Gamma_2 \Rightarrow P \).
- Local deduction theorem holds (\vdash):
  \( \Sigma \vdash \Gamma \cup \{Q\} \Rightarrow P \) if \( \Sigma \vdash \Gamma \Rightarrow Q \Rightarrow P \).

Soundness

**Theorem.** If \( \Sigma \vdash \Gamma \Rightarrow P \) holds, then \( \Sigma \models \Gamma \Rightarrow P \) holds.

**Proof.** Let \( \Sigma \vdash \Gamma \Rightarrow P \) hold.
- By compactness of \( \vdash \):
  there are finite sets \( \Sigma' \subseteq \Sigma \) and \( \Gamma' \subseteq \Gamma \) such that \( \Sigma' \vdash \Gamma' \Rightarrow P \) holds.
- By the local deduction theorem for \( \vdash \):
  \( \Sigma' \vdash \Gamma' \Rightarrow \phi_1 \to (\phi_2 \to \cdots \to (\phi_n \to P)\cdots) \).
- By the soundness of \( K \)-derivations:
  \( \Sigma' \models \Gamma' \Rightarrow \phi_1 \to (\phi_2 \to \cdots \to (\phi_n \to P)\cdots) \).
- By the local deduction theorem for \( \vdash \):
  \( \Sigma' \vdash \Gamma' \Rightarrow P \).
- By the monotonicity of \( \vdash \):
  \( \Sigma \models \Gamma \Rightarrow P \).

Completeness

**Theorem.** If \( \Sigma \models \Gamma \Rightarrow P \), then \( \Sigma \vdash \Gamma \Rightarrow P \).

**Proof.** Let \( \Sigma \models \Gamma \Rightarrow P \) hold.
- By compactness of \( \models \):
  there are finite sets \( \Sigma' \subseteq \Sigma \) and \( \Gamma' \subseteq \Gamma \) such that \( \Sigma' \vdash \Gamma' \Rightarrow P \) holds.
- By the local deduction theorem for \( \models \):
  \( \Sigma' \models \Gamma' \Rightarrow \phi_1 \to (\phi_2 \to \cdots \to (\phi_n \to P)\cdots) \).
- By the completeness of \( K \)-derivations:
  \( \Sigma' \vdash \Gamma' \Rightarrow \phi_1 \to (\phi_2 \to \cdots \to (\phi_n \to P)\cdots) \).
- By the local deduction theorem for \( \vdash \):
  \( \Sigma' \vdash \Gamma' \Rightarrow P \).
- By monotonicity of \( \vdash \):
  \( \Sigma \vdash \Gamma \Rightarrow P \).

5. Examples of Hilbert-style Proof Systems

- Using the proof system for \( K \) and formulas characterizing properties of frames we can construct Hilbert-style proof systems for other frame logics.
- As the first example we consider the modal logic \( T \) where the frames are reflexive.
- The characteristic formula for reflexive frames:
  \( T: \Box P \to P \).

**Proposition.** \( \Sigma \models \Gamma \Rightarrow P \) if \( \Sigma \cup [ T ] \models \Gamma \Rightarrow P \).

**Proposition.** \( \Sigma \models \Gamma \Rightarrow P \) if \( \Sigma \cup [ T ] \vdash \Gamma \Rightarrow P \).
**Modal Logic T**

Hence, a sound and complete Hilbert-style proof system for the modal logic T is obtained as follows:

**Classical axioms:** All tautologies

**Modal axioms:** All formulas of the form

- K: $\square (P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$
- T: $\square P \rightarrow P$

**Modus Ponens - rule**

N-rule

$\Rightarrow$

**Proposition.** $\Sigma \vdash T \Rightarrow P$ iff $\Sigma \vdash T \Rightarrow P$.

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**Modal Logic KD45**

KD45 is the collection of serial, transitive and euclidian frames.

**Proposition.** $\Sigma \vdash_{KD45} \Rightarrow P$ iff $\Sigma \vdash [\square] \cup [\square] \cup [\square] \vdash_{K} Y \Rightarrow P$.

A Hilbert-style proof system for KD45:

**Modal axioms:** All formulas of the form

- K: $\square (P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$
- D: $\square P \rightarrow \Diamond P$
- 4: $\square P \rightarrow \square \square P$
- 5: $\neg \square P \rightarrow \Diamond \neg \square P$

**Proposition.** $\Sigma \vdash_{KD45} \Rightarrow P$ iff $\Sigma \vdash_{KD45} \Rightarrow P$.

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**Modal Logic S5**

In a similar way for the frame logic S5 (equivalence frames):

**Proposition.** $\Sigma \vdash_{S5} \Rightarrow P$ iff $\Sigma \vdash [T] \cup [4] \cup [B] \vdash_{K} Y \Rightarrow P$ iff $\Sigma \vdash [T] \cup [4] \cup [5] \vdash_{K} Y \Rightarrow P$ iff $\Sigma \vdash [T] \cup [5] \vdash_{K} Y \Rightarrow P$.

A Hilbert-style proof system for the modal logic S5 (modal axioms need to extended):

**Modal axioms:** All formulas of the form

- K: $\square (P \rightarrow Q) \rightarrow (\square P \rightarrow \square Q)$
- T: $\square P \rightarrow P$
- 4: $\square P \rightarrow \square \square P$
- 5: $\neg \square P \rightarrow \Diamond \neg \square P$

**Proposition.** $\Sigma \vdash_{S5} \Rightarrow P$ iff $\Sigma \vdash_{S5} \Rightarrow P$.

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**Summary**

- A proof system of a logic is a syntactic calculus for showing that a formula is valid/a logical consequence from a set of formulas in the logic.
- For modal logics Hilbert-style axiomatic proof systems are common in the literature although they do not lend themselves well to automation.
- The two most important properties of a proof system are soundness and completeness.
- Typically soundness is quite straightforward to establish.
- For many frame logics completeness of Hilbert-style systems can be shown using the canonical model construction which is here demonstrated for the modal logic K.
- Using formulas characterizing properties of frames it is straightforward to construct Hilbert-style proof systems for many other frame logics.