PROOF THEORY FOR MODAL LOGICS

1. Hilbert-style proof theory
2. Soundness
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4. Generalization to local premises
5. Examples ($\mathbf{T}$, $\mathbf{S5}$ and $\mathbf{KD45}$)

M. Fitting: *Basic Modal Logic*, 1.7 (pp. 387 – 391).

Proof Systems

A proof system is a (syntactic) calculus to demonstrate that a given formula is valid/a logical consequence from a set of formulas.

A proof system gives a basis for developing automated reasoning techniques.

Possible proof systems:

- Axiomatic (Hilbert-style) proof theory
- Natural deduction
- Tableau methods
- Sequent calculus
- Resolution

1. Hilbert-style Proof Theory

For modal logic $\mathbf{K}$:

Classical axioms: All (classical) tautologies.

Modal axioms: All formulas of the form

$$\Box(P \to Q) \to (\Box P \to \Box Q)$$

Modus Ponens Rule: $\dfrac{P, P \to Q}{Q}$

Necessitation Rule (N-rule): $\dfrac{P}{\Box P}$

Derivation and Proof

(We first consider the case without any local premises).

Definition. A $\mathbf{K}$-derivation of a formula P from a set of formulas Σ is a finite sequence of formulas $\phi_1, \ldots, \phi_n$ such that $\phi_n = P$ and for all $i = 1, \ldots, n$

1. $\phi_i \in \Sigma$ or
2. ϕ_i is some axiom of $\mathbf{K}$ or
3. ϕ_i is obtained by one of the rules Modus Ponens or Necessitation from earlier formulas in the sequence.

Notation: $\Sigma \vdash_{\mathbf{K}} \emptyset \Longrightarrow P$ (or $\Sigma \vdash_{\mathbf{K}} P$)

A $\mathbf{K}$-proof of a formula P is a $\mathbf{K}$-derivation of P from the empty set of formulas.

Notation: $\emptyset \vdash_{\mathbf{K}} \emptyset \Longrightarrow P$ (or $\vdash_{\mathbf{K}} P$)

Example

A $\mathbf{K}$-proof for $\top \leftrightarrow \Box\top$:

1.	$\top$	(Taut)
2.	$\Box\top \rightarrow (\top \rightarrow \Box\top)$	(Taut)
3.	$\Box\top$	(N,1)
4.	$\top \rightarrow \Box\top$	(MP,2,3)
5.	$\top \rightarrow (\Box\top \rightarrow \top)$	(Taut)
6.	$\Box\top \rightarrow \top$	(MP,1,5)
7.	$(\Box\top \rightarrow \top) \rightarrow ((\top \rightarrow \Box\top) \rightarrow (\top \leftrightarrow \Box\top))$	(Taut)
8.	$(\top \rightarrow \Box\top) \rightarrow (\top \leftrightarrow \Box\top)$	(MP,6,7)
9.	$\top \leftrightarrow \Box\top$	(MP,4,8)

Derived Rules (I)

Regularity Rule for $\Box$ (R-rule): $\dfrac{P \rightarrow Q}{\Box P \rightarrow \Box Q}$

1.	$P \rightarrow Q$	
2.	$\Box(P \rightarrow Q)$	(N,1)
3.	$\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$	(K)
4.	$\Box P \rightarrow \Box Q$	(MP,2,3)

Generalized R-rule: $\dfrac{P_1 \wedge \cdots \wedge P_n \rightarrow Q}{\Box P_1 \wedge \cdots \wedge \Box P_n \rightarrow \Box Q}$

Derived Rules (II)

Regularity Rule for $\Diamond$ (R$\Diamond$-rule): $\dfrac{P \rightarrow Q}{\Diamond P \rightarrow \Diamond Q}$

1.	$P \rightarrow Q$	
2.	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$	(Taut)
3.	$\neg Q \rightarrow \neg P$	(MP,1,2)
4.	$\Box\neg Q \rightarrow \Box\neg P$	(R,3)
5.	$(\Box\neg Q \rightarrow \Box\neg P) \rightarrow (\neg\Box\neg P \rightarrow \neg\Box\neg Q)$	(Taut)
6.	$\neg\Box\neg P \rightarrow \neg\Box\neg Q$	(MP,4,5)

2. Soundness

The soundness of a proof system for a logic:

> If a formula is derivable in the proof system, it is also a logical consequence in the logic.

Theorem. If a formula P has a $\mathbf{K}$-derivation from a set of formulas Σ ($\Sigma \vdash_{\mathbf{K}} \emptyset \Longrightarrow P$), then $\Sigma \models_{\mathbf{K}} \emptyset \Longrightarrow P$
(or in short: if $\Sigma \vdash_{\mathbf{K}} P$, then $\Sigma \models_{\mathbf{K}} P$).

Proof.

Let $\phi_1, \ldots, \phi_n (= P)$ be a $\mathbf{K}$-derivation for a formula P.

We show by induction that for all $i = 1, \ldots, n$, $\Sigma \models_{\mathbf{K}} \phi_i$ holds (i.e., ϕ_i is valid in every model where Σ is valid).

So we prove by induction that for $i = 1, \ldots, n$, $\mathbf{C} \models \phi_i$ holds where $\mathbf{C} = \{M \mid M \models \Sigma\}$.

Induction Proof

- $(i=1)$: If $\phi_1 \in \Sigma$, clearly $\mathbf{C} \models \phi_1$ holds (by the definition of the collection of models $\mathbf{C}$). If ϕ_1 is a classical tautology or of the form $\Box(P \to Q) \to (\Box P \to \Box Q)$, then $\mathbf{C} \models \phi_1$ holds for every collection of models $\mathbf{C}$ by the basic theorem of the possible world semantic [ML-02].
- $(i>1)$: As above, if $\phi_i \in \Sigma$, is a classical tautology or of the form $\Box(P \to Q) \to (\Box P \to \Box Q)$, then $\mathbf{C} \models \phi_i$ holds.

  If ϕ_i is derived from earlier formulas in the proof by MP- or N-rules, by the inductive hypothesis the earlier formulas are $\mathbf{C}$-valid. As the set of $\mathbf{C}$-valid formulas is closed under MP- and N-rules [basic theorem of the possible world semantic], $\mathbf{C} \models \phi_i$ holds.

Hence, for all $i = 1,\ldots,n$, ϕ_i is $\mathbf{C}$-valid and thus $\Sigma \models_{\mathbf{K}} \phi_i$ holds. Hence, $\Sigma \models_{\mathbf{K}} P(=\phi_n)$ holds. ∎

3. Completeness

Completeness of a proof systems for a logic:

> If a formula is a logical consequence in the logic, then there is a derivation of it in the proof system.

Theorem. If $\Sigma \models_{\mathbf{K}} P$, then $\Sigma \vdash_{\mathbf{K}} P$.

The outline of the proof: Let $\Sigma \nvdash_{\mathbf{K}} P$ hold.

We show that then also $\Sigma \not\models_{\mathbf{K}} P$ holds.

This is done by constructing a canonical model $\mathcal{M}$ where all formulas in Σ are valid and for every Q such that $\Sigma \nvdash_{\mathbf{K}} Q$ holds there is a world s in $\mathcal{M}$ where $\mathcal{M}, s \not\Vdash Q$ holds.

The worlds of the model are maximally consistent sets of formulas that are constructed using Lindenbaum's lemma from the set of premises Σ.

(In)consistent Sets of Formulas

Definition. A finite set of formulas $\mathbf{A} = \{A_1,\ldots,A_n\}$ is said Σ-inconsistent if $\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1 \wedge \cdots \wedge A_n)$ holds.

Remark. The empty set is Σ-inconsistent if $\Sigma \vdash_{\mathbf{K}} \neg\top$ holds.

Definition. A set of formulas $\mathbf{A}$ is said Σ-consistent if none of its finite subsets is Σ-inconsistent.

As we assumed that the formula P has not $\mathbf{K}$-derivation from Σ, the set $\{\neg P\}$ is Σ-consistent.

This holds because the only other subset of $\{\neg P\}$, the empty set $\emptyset$, is also Σ-consistent which can be shown as follows: Assume that $\emptyset$ Σ-inconsistent, i.e., $\Sigma \vdash_{\mathbf{K}} \neg\top$ holds. Then $\Sigma \vdash_{\mathbf{K}} P$ holds as well because $\neg\top \to P$ is a tautology, a contradiction.

Important Lemmas

Lemma 1. If the set S is Σ-consistent, then each of its subsets $S' \subseteq S$ is Σ-consistent.

Proof. Assume that S has a subset S' which is not Σ-consistent. Then there is a Σ-inconsistent subset $A \subseteq S'$ but $A \subseteq S$ and, thus, S is not Σ-consistent, a contradiction. ∎

Lemma 2. If a set of formulas $\mathbf{A}$ is Σ-consistent and $\neg\Box Z \in \mathbf{A}$, then $\mathbf{A}^{\#} \cup \{\neg Z\}$ is also Σ-consistent where $\mathbf{A}^{\#} = \{Q \mid \Box Q \in \mathbf{A}\}$.

Proof. Assume that $\mathbf{A}^{\#} \cup \{\neg Z\}$ is Σ-inconsistent.

Then there is a set $\{A_1,\ldots,A_n\} \subseteq \mathbf{A}^{\#}$ such that $\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1 \wedge \cdots \wedge A_n \wedge \neg Z)$ holds.

(as $\neg(\top \wedge A_1 \wedge \cdots \wedge A_n) \to \neg(\top \wedge A_1 \wedge \cdots \wedge A_n \wedge \neg Z)$ is a tautology).

Proof cont'd:

1. $\neg(\top \wedge A_1 \wedge \cdots \wedge A_n \wedge \neg Z)$
2. $\neg\top \vee \neg A_1 \vee \cdots \vee \neg A_n \vee Z$ (Prop, 1)
3. $(\top \wedge A_1 \wedge \cdots \wedge A_n) \rightarrow Z$ (Prop, 2)
4. $(\Box\top \wedge \Box A_1 \wedge \cdots \wedge \Box A_n) \rightarrow \Box Z$ (GR, 3)
5. $\Box\top \rightarrow ((\Box A_1 \wedge \cdots \wedge \Box A_n) \rightarrow \Box Z)$ (Prop, 4)
6. $\top \rightarrow \Box\top$ (See. p. 5)
7. $\top \rightarrow ((\Box A_1 \wedge \cdots \wedge \Box A_n) \rightarrow \Box Z)$ (Prop, 5, 6)
8. $(\top \wedge \Box A_1 \wedge \cdots \wedge \Box A_n) \rightarrow \Box Z$ (Prop, 7)
9. $\neg(\top \wedge \Box A_1 \wedge \cdots \wedge \Box A_n \wedge \neg\Box Z)$ (Prop, 8)

$\Longrightarrow$ $\mathbf{A}$ is Σ-inconsistent (a contradiction). Hence, $\mathbf{A}^{\#} \cup \{\neg Z\}$ Σ-consistent. ■

Lindenbaum's Lemma

Definition. Γ is maximally Σ-consistent if Γ is Σ-consistent and all supersets $\Gamma' \supset \Gamma$ are Σ-inconsistent.

Lemma 3. (Lindenbaum) Every Σ-consistent set of formulas can be extended to a maximally Σ-consistent one.

Proof. Let $\mathbf{A}$ be Σ-consistent. Enumerate all modal formulas in a sequence $Q_0, Q_1, \ldots$ and define set $\Delta_0, \Delta_1, \ldots$ and Δ as follows:

$\Delta_0 = \mathbf{A}$.

$$\Delta_i = \begin{cases} \Delta_{i-1} \cup \{Q_{i-1}\} & \text{if } \Delta_{i-1} \cup \{Q_{i-1}\} \ \Sigma\text{-consistent} \\ \Delta_{i-1} \cup \{\neg Q_{i-1}\} & \text{otherwise} \end{cases}$$

$$\Delta = \bigcup_{i \geq 0} \Delta_i$$

We establish properties (i-iv) which imply the lemma.

(i) $\mathbf{A} \subseteq \Delta$

(ii) for all $i = 0, 1, \ldots$, the set Δ_i is Σ-consistent.

Δ_0 is Σ-consistent.

Let Δ_{i-1} be Σ-consistent.

Assume that Δ_i is Σ-inconsistent. Then $\Delta_i = \Delta_{i-1} \cup \{\neg Q_{i-1}\}$ and $\Delta_{i-1} \cup \{Q_{i-1}\}$ are Σ-inconsistent.

Hence, there is a set $\{A_1^+, \ldots, A_{n^+}^+\} \subseteq \Delta_{i-1}$ such that

$$\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+ \wedge Q_{i-1})$$

and a set $\{A_1^-, \ldots, A_{n^-}^-\} \subseteq \Delta_{i-1}$ such that

$$\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^- \wedge \neg Q_{i-1}).$$

We can continue the derivations of the two formulas above:

1. $\neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+ \wedge Q_{i-1})$
2. $\neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^- \wedge \neg Q_{i-1})$
3. $Q_{i-1} \rightarrow \neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+)$ (Prop, 1)
4. $\neg Q_{i-1} \rightarrow \neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^-)$ (Prop, 2)
5. $\neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+) \vee$
   $\neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^-)$ (Prop, 3, 4)
6. $\neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+ \wedge A_1^- \wedge \cdots \wedge A_{n^-}^-)$ (Prop, 5)

$\Longrightarrow$ Δ_{i-1} is Σ-inconsistent, a contradiction.

(iii) Δ is Σ-consistent.

Assume that Δ is Σ-inconsistent.

Hence, there is a set $\{A_1,\ldots,A_n\} \subseteq \Delta$ such that

$$\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1 \wedge \cdots \wedge A_n).$$

So there is $i \geq 0$ such that $\{A_1,\ldots,A_n\} \subseteq \Delta_i$.

$\Longrightarrow$ Δ_i is Σ-inconsistent, a contradiction.

(iv) Δ is maximally Σ-consistent.

Assume that $\Delta \cup \{Z\}$ Σ-consistent for some $Z \notin \Delta$.

As $Z = Q_i$ for some i, $\Delta \cup \{Q_i\}$ is Σ-consistent.

Because $\Delta_i \cup \{Q_i\} \subseteq \Delta \cup \{Q_i\}$ also $\Delta_i \cup \{Q_i\}$ is Σ-consistent [Lemma 1].

Thus, $Z \in \Delta_{i+1} \subseteq \Delta$, a contradiction.

(i-iv) $\Longrightarrow$ Δ is maximally Σ-consistent extension of the set **A**. ■

Important Lemmas—cont'd

Lemma 4. For all maximally Σ-consistent sets Γ,
(i) $\Sigma \subseteq \Gamma$ and (ii) either $Z \in \Gamma$ or $\neg Z \in \Gamma$ for all formulas Z.

Proof. (i) Assume that $Z \in \Sigma - \Gamma$. Then $\Gamma \cup \{Z\}$ is Σ-inconsistent and there is a set $\{A_1,\ldots,A_n\} \subseteq \Gamma$ such that $\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1 \wedge \cdots \wedge A_n \wedge Z)$.
We can continue the derivation:

1.	$\neg(\top \wedge A_1 \wedge \cdots \wedge A_n \wedge Z)$	
2.	$Z \rightarrow \neg(\top \wedge A_1 \wedge \cdots \wedge A_n)$	(Prop, 1)
3.	Z	(GP)
4.	$\neg(\top \wedge A_1 \wedge \cdots \wedge A_n)$	(MP, 2, 3)

Hence, $\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1 \wedge \cdots \wedge A_n)$ holds.

$\Longrightarrow$ Γ is Σ-inconsistent, a contradiction. Hence, $\Sigma \subseteq \Gamma$ holds.

(ii) Because Γ is Σ-consistent, $\{Z, \neg Z\} \not\subseteq \Gamma$ ($\neg(\top \wedge Z \wedge \neg Z)$ is a tautology). Assume that $Z \notin \Gamma$ and $\neg Z \notin \Gamma$.

Then there is a set $\{A_1^+,\ldots,A_{n^+}^+\} \subseteq \Gamma$ such that
$\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+ \wedge Z)$
and a set $\{A_1^-,\ldots,A_{n^-}^-\} \subseteq \Gamma$ such that $\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^- \wedge \neg Z)$.

We can continue the two derivations:

1.	$\neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+ \wedge Z)$	
2.	$\neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^- \wedge \neg Z)$	
3.	$Z \rightarrow \neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+)$	(Prop, 1)
4.	$\neg Z \rightarrow \neg(\top \wedge A_1^- \wedge \cdots \wedge A_{n^-}^-)$	(Prop, 2)
5.	$\neg(\top \wedge A_1^+ \wedge \cdots \wedge A_{n^+}^+ \wedge A_1^- \wedge \cdots \wedge A_{n^-}^-)$	(Prop, 3, 4)

$\Longrightarrow$ Γ is Σ-inconsistent, a contradiction.

Hence, either $Z \in \Gamma$ or $\neg Z \in \Gamma$ for all formulas Z. ■

Canonical Model

We construct the canonical model $\mathcal{M} = \langle S, R, v \rangle$ as follows:

- S is the collection of all maximally Σ-consistent sets.
- For all $s,t \in S$: sRt iff $s^\# \subseteq t$.
- For all atomic propositions Q: $v(s,Q) = \text{true}$ iff $Q \in s$.

Lemma 5. For all formulas Q, for all $s \in S$ it holds that

$$\mathcal{M}, s \Vdash Q \text{ iff } Q \in s.$$

Proof. We use structural induction.

- The formula $\bot$: $\mathcal{M}, s \nVdash \bot$.

  Assume that $\bot \in s$. Because $\Sigma \vdash_{\mathbf{K}} \neg(\top \wedge \bot)$, the set of formulas s is Σ-inconsistent, a contradiction. Hence, $\bot \notin s$ holds.

- For atomic propositions Q the claim holds by the definition of $\mathcal{M}$.
- For a formula of the form $\neg Q$: $\mathcal{M},s \Vdash \neg Q$ iff $\mathcal{M},s \nVdash Q$ iff [IH] $Q \notin s$ iff [Lemma 4 (ii)] $\neg Q \in s$.
- For a formula of the form $Q \to P$ the claim can be proved as above.
- For a formula of the form $\Box Q$:
  ($\Leftarrow$) Let $\Box Q \in s$ hold. If sRt, then $s^{\#} \subseteq t$, $Q \in t$ and $\mathcal{M},t \Vdash Q$ [IH]. Thus, $\mathcal{M},s \Vdash \Box Q$.
  ($\Rightarrow$) Let $\Box Q \notin s$ hold. Then $\neg\Box Q \in s$ [Lemma 4 (ii)].
  Now $t_0 = s^{\#} \cup \{\neg Q\}$ is Σ-consistent [Lemma 2] and t_0 has a maximally Σ-consistent extension t [Lemma 3 (Lindenbaum)].
  So sRt because $s^{\#} \subseteq t$. As $\neg Q \in t_0 \subseteq t$, $Q \notin t$ holds [Lemma 4 (ii)].
  Hence, $\mathcal{M},t \nVdash Q$ [IH] and $\mathcal{M},s \nVdash \Box Q$.

■

Completeness Proof—Summary

- Because $\Sigma \subseteq s$ for all $s \in S$ [Lemma 4 (i)], the set Σ is valid in the canonical model $\mathcal{M}$ [Lemma 5].
- As the set $\{\neg P\}$ is Σ-consistent, the set has a maximal Σ-consistent extension $t \in S$ [Lemma 3] and $P \notin t$ [Lemma 4 (ii)].
  Thus, $\mathcal{M},t \nVdash P$ [Lemma 5].
- As Σ is valid in $\mathcal{M} = \langle S,R,v\rangle$ and there is a world $t \in S$ such that $\mathcal{M},t \nVdash P$ holds, also $\Sigma \not\models_{\mathbf{K}} P$ holds.

$\Longrightarrow$ Hilbert-style proof theory for the modal logic $\mathbf{K}$ is complete.

4. Generalization to Local Premises

Definition. $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ means that there is a sequence of formulas ending with P consisting of a global part, coming first, and a local part, coming last.

In the global part every formula is

- an axiom of $\mathbf{K}$, belongs to the set Σ or
- is obtained by one of the rules Modus Ponens or Necessitation from earlier formulas in the sequence.

In the local part every formula is

- an axiom of $\mathbf{K}$, belongs to the set Υ or
- is obtained by the Modus Ponens rule from earlier formulas in the sequence.

Example

We show that $\{P,\ P \to Q\} \vdash_{\mathbf{K}} \{\Box Q \to R\} \Longrightarrow R$ holds:

1.	P	(GP)
2.	$P \to Q$	(GP)
3.	Q	(MP,1,2)
4.	$\Box Q$	(N,3)
5.	$\Box Q \to R$	(LP)
6.	R	(MP,4,5)

Note that the N-rule cannot be used in the local part.

For example, $\{P, P \to Q\} \vdash_{\mathbf{K}} \{\Box Q \to R\} \Longrightarrow \Box R$ does not hold.

Properties of Derivations

- Derivations are finite.

  $\Longrightarrow$ *Compactness* ($\vdash$):

  If $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ holds, then there are finite sets $\Sigma' \subseteq \Sigma$ and $\Upsilon' \subseteq \Upsilon$ such that $\Sigma' \vdash_{\mathbf{K}} \Upsilon' \Longrightarrow P$ holds.

- MP- and N-rules are monotonic:

  $\Longrightarrow$ *Monotonicity* ($\vdash$):

  Let $\Sigma_1 \subseteq \Sigma_2$ and $\Upsilon_1 \subseteq \Upsilon_2$ hold. Then if $\Sigma_1 \vdash_{\mathbf{K}} \Upsilon_1 \Longrightarrow P$, then $\Sigma_2 \vdash_{\mathbf{K}} \Upsilon_2 \Longrightarrow P$.

- *Local deduction theorem* holds ($\vdash$):

  $\Sigma \vdash_{\mathbf{K}} \Upsilon \cup \{Q\} \Longrightarrow P$ iff $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow Q \rightarrow P$.

Completeness

Theorem. If $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$, then $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Proof. Let $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ hold.

- By compactness of $\models$:
  there are finite sets $\Sigma' \subseteq \Sigma$ and $\Upsilon' = \{\phi_1, \ldots, \phi_n\} \subseteq \Upsilon$ such that $\Sigma' \models_{\mathbf{K}} \Upsilon' \Longrightarrow P$ holds..

- By the local deduction theorem for $\models$:
  $\Sigma' \models_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow P) \cdots)$.

- By the completeness of $\mathbf{K}$-derivations:
  $\Sigma' \vdash_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \rightarrow (\phi_2 \cdots \rightarrow (\phi_n \rightarrow P) \cdots)$.

- By the local deduction theorem for $\vdash$:
  $\Sigma' \vdash_{\mathbf{K}} \Upsilon' \Longrightarrow P$.

- By monotonicity of $\vdash$:
  $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$.

■

Soundness

Theorem. If $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ holds, then $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ holds.

Proof. Let $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ hold.

- By compactness of $\vdash$:
  there are finite sets $\Sigma' \subseteq \Sigma$ and $\Upsilon' = \{\phi_1, \ldots, \phi_n\} \subseteq \Upsilon$ such that $\Sigma' \vdash_{\mathbf{K}} \Upsilon' \Longrightarrow P$.

- By the local deduction theorem for $\vdash$:
  $\Sigma' \vdash_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \rightarrow (\phi_2 \rightarrow \cdots \rightarrow (\phi_n \rightarrow P) \cdots)$.

- By the soundness of $\mathbf{K}$-derivations:
  $\Sigma' \models_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \rightarrow (\phi_2 \cdots \rightarrow (\phi_n \rightarrow P) \cdots)$.

- By the local deduction theorem for $\models$:
  $\Sigma' \models_{\mathbf{K}} \Upsilon' \Longrightarrow P$.

- By the monotonicity of $\models$:
  $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

■

5. Examples of Hilbert-style Proof Systems

- Using the proof system for $\mathbf{K}$ and formulas characterizing properties of frames we can construct Hilbert-style proof systems for other frame logics.

- As the first example we consider the modal logic $\mathbf{T}$ where the frames are reflexive.

- The characteristic formula for reflexive frames:
  T: $\Box P \rightarrow P$.

Proposition. $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{T}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

$\Longrightarrow$ (Soundness and completeness of $\mathbf{K}$-derivations)

Proposition. $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{T}]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Modal Logic T

Hence, a sound and complete Hilbert-style proof system for the modal logic $\mathbf{T}$ is obtained as follows:

Classical axioms: All tautologies

Modal axioms: All formulas of the form

K: $\Box(P \to Q) \to (\Box P \to \Box Q)$

T: $\Box P \to P$

Modus Ponens -rule

N-rule

$\Longrightarrow$

Proposition. $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$ iff $\Sigma \vdash_{\mathbf{T}} \Upsilon \Longrightarrow P$.

Modal Logic S5

In a similar way for the frame logic $\mathbf{S5}$ (equivalence frames):

Proposition. $\Sigma \models_{\mathbf{S5}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![T]\!] \cup [\![4]\!] \cup [\![B]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![T]\!] \cup [\![4]\!] \cup [\![5]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![T]\!] \cup [\![5]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$.

A Hilbert-style proof system for the modal logic $\mathbf{S5}$ (modal axioms need to extended):

Modal axioms: All formulas of the form

K: $\Box(P \to Q) \to (\Box P \to \Box Q)$

T: $\Box P \to P$

$4: \Box P \to \Box\Box P$

$5: \neg\Box P \to \Box\neg\Box P$

Proposition. $\Sigma \models_{\mathbf{S5}} \Upsilon \Longrightarrow P$ iff $\Sigma \vdash_{\mathbf{S5}} \Upsilon \Longrightarrow P$.

Modal Logic KD45

$\mathbf{KD45}$ is the collection of serial, transitive and eudlidian frames.

Proposition. $\Sigma \models_{\mathbf{KD45}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![D]\!] \cup [\![4]\!] \cup [\![5]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$.

A Hilbert-style proof system for $\mathbf{KD45}$:

Modal axioms: All formulas of the form

K: $\Box(P \to Q) \to (\Box P \to \Box Q)$

$D: \Box P \to \Diamond P$

$4: \Box P \to \Box\Box P$

$5: \neg\Box P \to \Box\neg\Box P$

Proposition. $\Sigma \models_{\mathbf{KD45}} \Upsilon \Longrightarrow P$ iff $\Sigma \vdash_{\mathbf{KD45}} \Upsilon \Longrightarrow P$.

Summary

- A proof system of a logic is a syntactic calculus for showing that a formula is valid/a logical consequence from a set of formulas in the logic.
- For modal logics Hilbert-style axiomatic proof systems are common in the literature although they do not lend themselves well to automation.
- The two most important properties of a proof system are soundness and completeness.
- Typically soundness is quite straightforward to establish.
- For many frame logics completeness of Hilbert-style systems can be shown using the canonical model construction which is here demonstrated for the modal logic $\mathbf{K}$.
- Using formulas characterizing properties of frames it is straightforward to construct Hilbert-style proof systems for many other frame logics.