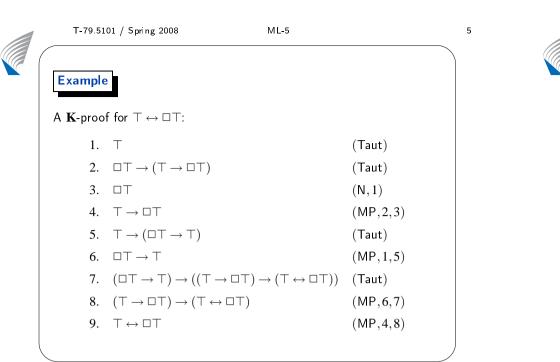
| PROOF THEORY FOR MODAL LOGICS   | 1. Hilbert-style Proof Theory  |
|---|--|
| 1. Hilbert-style proof theory   | For modal logic <b>K</b> :   |
| 2. Soundness  | Classical axioms: All (classical) tautologies.   |
| 3. Completeness   | Modal axioms: All formulas of the form   |
| 4. Generalization to local premises   | $\Box(P  ightarrow Q)  ightarrow (\Box P  ightarrow \Box Q)$   |
| 5. Examples (T, S5 and KD45)  | Modus Ponens Rule: $\frac{P, P \rightarrow Q}{Q}$  |
| M. Fitting: <i>Basic Modal Logic</i> , 1.7 (pp. 387 – 391).   | Necessitation Rule (N-rule): $\frac{P}{\Box P}$  |
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|   | T-79.5101 / Spring 2008 ML-5   |
| T-79.5101 / Spring 2008 ML-5 2  | T-79.5101 / Spring 2008 ML-5<br>Derivation and Proof<br>(We first consider the case without any local premises).<br>Definition. A K-derivation of a formula P from a set of formulas Σ is  |
| T-79.5101 / Spring 2008 ML-5 2           Proof Systems           A proof system is a (syntactic) calculus to demonstrate that a given   | T-79.5101 / Spring 2008 ML-5           Derivation and Proof           (We first consider the case without any local premises).   |
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| T-79.5101 / Spring 2008 ML-5 2<br>Proof Systems<br>A proof system is a (syntactic) calculus to demonstrate that a given<br>formula is valid/a logical consequence from a set of formulas.<br>A proof system gives a basis for developing automated reasoning<br>techniques.   | T-79.5101 / Spring 2008 ML-5<br>$\begin{array}{c} \hline \\ \hline $   |
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| T-79.5101 / Spring 2008 ML-5 2<br>Proof Systems<br>A proof system is a (syntactic) calculus to demonstrate that a given<br>formula is valid/a logical consequence from a set of formulas.<br>A proof system gives a basis for developing automated reasoning<br>techniques.<br>Possible proof systems:<br>• Axiomatic (Hilbert-style) proof theory                        | T-79.5101 / Spring 2008 ML-5<br>$\begin{array}{c} \hline \\ \hline $   |
| T-79.5101 / Spring 2008 ML-5 2<br>Proof Systems<br>A proof system is a (syntactic) calculus to demonstrate that a given<br>formula is valid/a logical consequence from a set of formulas.<br>A proof system gives a basis for developing automated reasoning<br>techniques.<br>Possible proof systems:<br>• Axiomatic (Hilbert-style) proof theory<br>• Natural deduction | <ul> <li>T-79.5101 / Spring 2008 ML-5</li> <li>Derivation and Proof</li> <li>(We first consider the case without any local premises).</li> <li>Definition. A K-derivation of a formula P from a set of formulas Σ is a finite sequence of formulas φ<sub>1</sub>,,φ<sub>n</sub> such that φ<sub>n</sub> = P and for all i = 1,,n</li> <li>φ<sub>i</sub> ∈ Σ or</li> <li>φ<sub>i</sub> is some axiom of K or</li> <li>φ<sub>i</sub> is obtained by one of the rules Modus Ponens or Necessitation from earlier formulas in the sequence.</li> </ul> |



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 $P \rightarrow Q$   $P \rightarrow Q$  (N, 1)  $P \rightarrow Q$  (N, 2, 3)  $Reneralized R-rule: \frac{P_1 \wedge \cdots \wedge P_n \rightarrow Q}{\Box P_1 \wedge \cdots \wedge \Box P_n \rightarrow \Box Q}$ 

#### Derived Rules (II)

Regularity Rule for 
$$\diamond$$
 (R $\diamond$ -rule):  $\frac{P \rightarrow Q}{\diamond P \rightarrow \diamond Q}$ 

- 1.  $P \rightarrow Q$ 2.  $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$  (Taut) 3.  $\neg Q \rightarrow \neg P$  (MP, 1, 2) 4.  $\Box \neg Q \rightarrow \Box \neg P$  (R, 3) 5.  $(\Box \neg Q \rightarrow \Box \neg P) \rightarrow (\neg \Box \neg P \rightarrow \neg \Box \neg Q)$  (Taut) 6.  $\neg \Box \neg P \rightarrow \neg \Box \neg Q$  (MP, 4, 5)
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T-79 5101 / Spring 2008 ML-5 8 2. Soundness The soundness of a proof system for a logic: If a formula is derivable in the proof system, it is also a logical consequence in the logic. **Theorem.** If a formula *P* has a **K**-derivation from a set of formulas  $\Sigma$  $(\Sigma \vdash_{\mathbf{K}} \emptyset \Longrightarrow P)$ , then  $\Sigma \models_{\mathbf{K}} \emptyset \Longrightarrow P$ (or in short: if  $\Sigma \vdash_{\mathbf{K}} P$ , then  $\Sigma \models_{\mathbf{K}} P$ ). Proof. Let  $\phi_1, \ldots, \phi_n (= P)$  be a **K**-derivation for a formula *P*. We show by induction that for all i = 1, ..., n,  $\Sigma \models_{\mathbf{K}} \phi_i$  holds (i.e.,  $\phi_i$  is valid in every model where  $\Sigma$  is valid). So we prove by induction that for i = 1, ..., n,  $\mathbf{C} \models \phi_i$  holds where  $\mathbf{C} = \{ M \mid M \models \Sigma \}.$ 

#### Induction Proof

- (i = 1): If  $\phi_1 \in \Sigma$ , clearly  $\mathbf{C} \models \phi_1$  holds (by the definition of the collection of models  $\mathbf{C}$ ). If  $\phi_1$  is a classical tautology or of the form  $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$ , then  $\mathbf{C} \models \phi_1$  holds for every collection of models  $\mathbf{C}$  by the basic theorem of the possible world semantic [ML-02].
- (i > 1): As above, if  $\phi_i \in \Sigma$ , is a classical tautology or of the form  $\Box(P \to Q) \to (\Box P \to \Box Q)$ , then  $\mathbb{C} \models \phi_i$  holds.

If  $\phi_i$  is derived from earlier formulas in the proof by MP- or N-rules, by the inductive hypothesis the earlier formulas are C-valid. As the set of C-valid formulas is closed under MP- and N-rules [basic theorem of the possible world semantic],  $\mathbf{C} \models \phi_i$ holds.

Hence, for all i = 1, ..., n,  $\phi_i$  is C-valid and thus  $\Sigma \models_{\mathbf{K}} \phi_i$  holds. Hence,  $\Sigma \models_{\mathbf{K}} P(=\phi_n)$  holds.

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# 3. Completeness

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#### Completeness of a proof systems for a logic:

If a formula is a logical consequence in the logic, then there is a derivation of it in the proof system.

**Theorem.** If  $\Sigma \models_{\mathbf{K}} P$ , then  $\Sigma \vdash_{\mathbf{K}} P$ .

```
The outline of the proof: Let \Sigma \not\vdash_{\mathbf{K}} P hold.
```

We show that then also  $\Sigma \not\models_{\mathbf{K}} P$  holds.

This is done by constructing a canonical model  $\mathcal{M}$  where all formulas in  $\Sigma$  are valid and for every Q such that  $\Sigma \not\vdash_{\mathbf{K}} Q$  holds there is a world s in  $\mathcal{M}$  where  $\mathcal{M}, s \mid \not\vdash Q$  holds.

The worlds of the model are maximally consistent sets of formulas that are constructed using Lindenbaum's lemma from the set of premises  $\Sigma.$ 

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## (In)consistent Sets of Formulas

**Definition.** A finite set of formulas  $\mathbf{A} = \{A_1, \dots, A_n\}$  is said  $\Sigma$ -inconsistent if  $\Sigma \vdash_{\mathbf{K}} \neg (\top \land A_1 \land \dots \land A_n)$  holds.

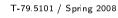
**Remark.** The empty set is  $\Sigma$ -inconsistent if  $\Sigma \vdash_{\mathbf{K}} \neg \top$  holds.

**Definition.** A set of formulas **A** is said  $\Sigma$ -consistent if none of its finite subsets is  $\Sigma$ -inconsistent.

As we assumed that the formula P has not **K**-derivation from  $\Sigma$ , the set  $\{\neg P\}$  is  $\Sigma$ -consistent.

This holds because the only other subset of  $\{\neg P\}$ , the empty set  $\emptyset$ , is also  $\Sigma$ -consistent which can be shown as follows: Assume that  $\emptyset$  $\Sigma$ -inconsistent, i.e.,  $\Sigma \vdash_{\mathbf{K}} \neg \top$  holds. Then  $\Sigma \vdash_{\mathbf{K}} P$  holds as well because  $\neg \top \rightarrow P$  is a tautology, a contradiction.

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#### Important Lemmas

**Lemma 1.** If the set S is  $\Sigma$ -consistent, then each of its subsets  $S' \subseteq S$  is  $\Sigma$ -consistent.

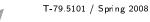
**Proof.** Assume that S has a subset S' which is not  $\Sigma$ -consistent. Then there is a  $\Sigma$ -inconsistent subset  $A \subseteq S'$  but  $A \subseteq S$  and, thus, S is not  $\Sigma$ -consistent, a contradiction.

**Lemma 2.** If a set of formulas **A** is  $\Sigma$ -consistent and  $\neg \Box Z \in \mathbf{A}$ , then  $\mathbf{A}^{\#} \cup \{\neg Z\}$  is also  $\Sigma$ -consistent where  $\mathbf{A}^{\#} = \{Q \mid \Box Q \in \mathbf{A}\}$ .

**Proof.** Assume that  $\mathbf{A}^{\#} \cup \{\neg Z\}$  is  $\Sigma$ -inconsistent.

Then there is a set  $\{A_1, \ldots, A_n\} \subseteq \mathbf{A}^{\#}$  such that  $\Sigma \vdash_{\mathbf{K}} \neg (\top \land A_1 \land \cdots \land A_n \land \neg Z)$  holds.

(as  $\neg(\top \land A_1 \land \cdots \land A_n) \rightarrow \neg(\top \land A_1 \land \cdots \land A_n \land \neg Z)$  is a tautology).



Proof cont'd:

| Proof cont d:   |   |              |  |
|---|---|--------------|--|
| 1.  | $\neg(\top \land A_1 \land \cdots \land A_n \land \neg Z)$                |              |  |
| 2.  | $\neg \top \lor \neg A_1 \lor \cdots \lor \neg A_n \lor Z$                | (Prop,1)     |  |
| 3.  | $(\top \wedge A_1 \wedge \cdots \wedge A_n) \to Z$                        | (Prop,2)     |  |
| 4.  | $(\Box \top \land \Box A_1 \land \cdots \land \Box A_n) \to \Box Z$       | (GR,3)       |  |
| 5.  | $\Box \top \to ((\Box A_1 \land \cdots \land \Box A_n) \to \Box Z)$       | (Prop, 4)    |  |
| 6.  | $\top \to \Box \top$  | (See. p. 5)  |  |
| 7.  | $\top \to ((\Box A_1 \land \cdots \land \Box A_n) \to \Box Z)$            | (Prop, 5, 6) |  |
| 8.  | $(\top \land \Box A_1 \land \cdots \land \Box A_n) \to \Box Z$            | (Prop,7)     |  |
| 9.  | $\neg(\top \land \Box A_1 \land \cdots \land \Box A_n \land \neg \Box Z)$ | (Prop, 8)    |  |
| $\implies$ <b>A</b> is $\Sigma$ -inconsistent (a contradiction). Hence, $\mathbf{A}^{\#} \cup \{\neg Z\}$ |   |              |  |
| $\Sigma$ -consistent.   |   |              |  |
|   |   |              |  |

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# Lindenbaum's Lemma

**Definition**.  $\Gamma$  is maximally  $\Sigma$ -consistent if  $\Gamma$  is  $\Sigma$ -consistent and all supersets  $\Gamma' \supset \Gamma$  are  $\Sigma$ -inconsistent.

**Lemma 3.** (Lindenbaum) Every  $\Sigma$ -consistent set of formulas can be extended to a maximally  $\Sigma$ -consistent one.

**Proof.** Let **A** be  $\Sigma$ -consistent. Enumerate all modal formulas in a sequence  $Q_0, Q_1, \ldots$  and define set  $\Delta_0, \Delta_1, \ldots$  and  $\Delta$  as follows:

$$\begin{split} \Delta_0 &= \mathbf{A}. \\ \Delta_i &= \begin{cases} \Delta_{i-1} \cup \{Q_{i-1}\} & \text{if } \Delta_{i-1} \cup \{Q_{i-1}\} \text{ } \Sigma\text{-consistent} \\ \Delta_{i-1} \cup \{\neg Q_{i-1}\} & \text{otherwise} \end{cases} \\ \Delta &= \bigcup_{i \geq 0} \Delta_i \end{split}$$

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We establish properties (i-iv) which imply the lemma.

(i)  $\mathbf{A} \subseteq \Delta$ 

- (ii) for all i = 0, 1, ..., the set  $\Delta_i$  is  $\Sigma$ -consistent.
  - $\Delta_0$  is  $\Sigma$ -consistent.
  - Let  $\Delta_{i-1}$  be  $\Sigma$  consistent.

Assume that  $\Delta_i$  is  $\Sigma$ -inconsistent. Then  $\Delta_i = \Delta_{i-1} \cup \{\neg Q_{i-1}\}$  and  $\Delta_{i-1} \cup \{Q_{i-1}\}$  are  $\Sigma$ -inconsistent.

Hence, there is a set  $\{A_1^+,\ldots,A_{n^+}^+\}\subseteq\Delta_{i-1}$  such that

$$\begin{split} \Sigma \vdash_{\mathbf{K}} \neg (\top \wedge A_{1}^{+} \wedge \cdots \wedge A_{n^{+}}^{+} \wedge Q_{i-1}) \\ \text{and a set } \{A_{1}^{-}, \dots, A_{n^{-}}^{-}\} \subseteq \Delta_{i-1} \text{ such that} \\ \Sigma \vdash_{\mathbf{K}} \neg (\top \wedge A_{1}^{-} \wedge \cdots \wedge A_{n^{-}}^{-} \wedge \neg Q_{i-1}). \end{split}$$

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We can continue the derivations of the two formulas above:

 $\implies \Delta_{i-1}$  is  $\Sigma$ -inconsistent, a contradiction.

(iii)  $\Delta$  is  $\Sigma$ -consistent.

[Lemma 1]

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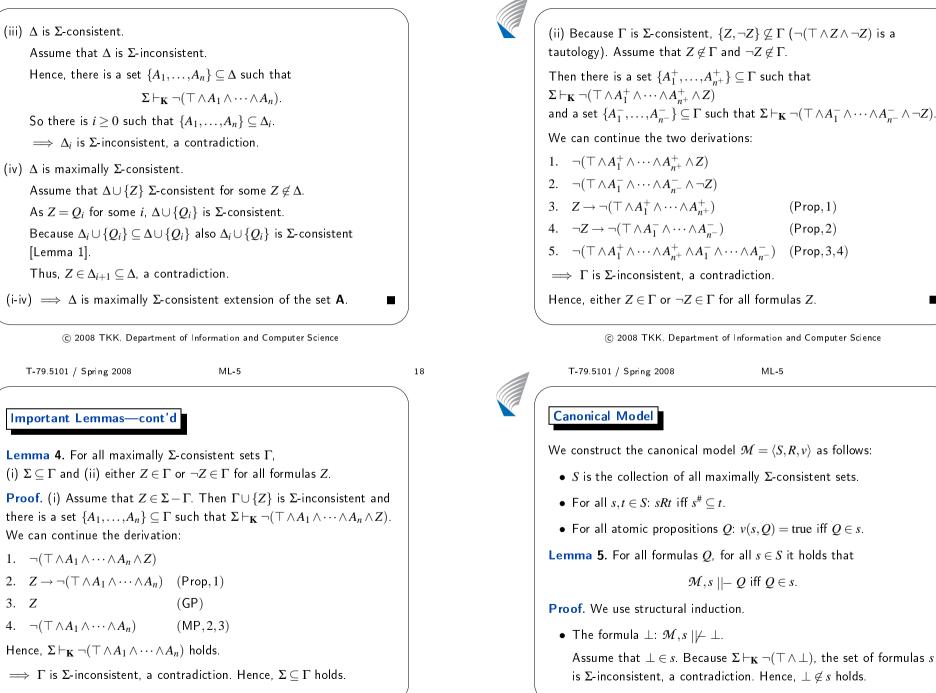
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- For atomic propositions Q the claim holds by the definition of  $\mathcal{M}.$
- For a formula of the form  $\neg Q$ :  $\mathcal{M}, s \mid \models \neg Q$  iff  $\mathcal{M}, s \mid \models Q$  iff [IH]  $Q \notin s$  iff [Lemma 4 (ii)]  $\neg Q \in s$ .
- For a formula of the form  $Q \to P$  the claim can be proved as above.
- For a formula of the form □Q: (⇐) Let □Q ∈ s hold. If sRt, then s<sup>#</sup> ⊆ t, Q ∈ t and M,t ||-Q [IH]. Thus, M,s ||- □Q.
  (⇒) Let □Q ∉ s hold. Then ¬□Q ∈ s [Lemma 4 (ii)]. Now t<sub>0</sub> = s<sup>#</sup> ∪ {¬Q} is Σ-consistent [Lemma 2] and t<sub>0</sub> has a maximally Σ-consistent extension t [Lemma 3 (Lindenbaum)]. So sRt because s<sup>#</sup> ⊆ t. As ¬Q ∈ t<sub>0</sub> ⊆ t, Q ∉ t holds [Lemma 4 (ii)]. Hence, M,t ||-Q [IH] and M,s ||-□Q.

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#### Completeness Proof—Summary

- Because Σ ⊆ s for all s ∈ S [Lemma 4 (i)], the set Σ is valid in the canonical model *M* [Lemma 5].
- As the set  $\{\neg P\}$  is  $\Sigma$ -consistent, the set has a maximal  $\Sigma$ -consistent extension  $t \in S$  [Lemma 3] and  $P \notin t$  [Lemma 4 (ii)]. Thus,  $\mathcal{M}, t \models P$  [Lemma 5].
- As  $\Sigma$  is valid in  $\mathcal{M} = \langle S, R, v \rangle$  and there is a world  $t \in S$  such that  $\mathcal{M}, t \mid \not\vdash P$  holds, also  $\Sigma \not\models_{\mathbf{K}} P$  holds.
- $\implies$  Hilbert-style proof theory for the modal logic K is complete.

## 4. Generalization to Local Premises

**Definition**.  $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$  means that there is a sequence of formulas ending with P consisting of a global part, coming first, and a local part, coming last.

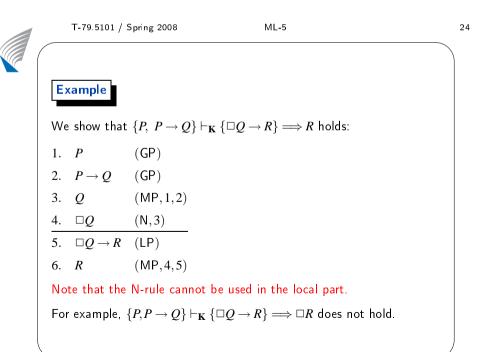
In the global part every formula is

- an axiom of **K**, belongs to the set  $\Sigma$  or
- is obtained by one of the rules Modus Ponens or Necessitation from earlier formulas in the sequence.

In the local part every formula is

- an axiom of K, belongs to the set  $\Upsilon$  or
- is obtained by the Modus Ponens rule from earlier formulas in the sequence.

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**Properties of Derivations** 

- Derivations are finite.
- $\implies Compactness (\vdash):$ If  $\Sigma \vdash_{\mathbf{K}} \Upsilon \implies P$  holds, then there are finite sets  $\Sigma' \subseteq \Sigma$  and  $\Upsilon' \subseteq \Upsilon$

such that  $\Sigma' \vdash_{\mathbf{K}} \Upsilon' \Longrightarrow P$  holds.

• MP- and N-rules are monotonic:

 $\implies$  *Monotonicity* ( $\vdash$ ):

Let  $\Sigma_1 \subseteq \Sigma_2$  and  $\Upsilon_1 \subseteq \Upsilon_2$  hold. Then if  $\Sigma_1 \vdash_{\mathbf{K}} \Upsilon_1 \Longrightarrow P$ , then  $\Sigma_2 \vdash_{\mathbf{K}} \Upsilon_2 \Longrightarrow P$ .

• *Local deduction theorem* holds (⊢):

$$\Sigma \vdash_{\mathbf{K}} \Upsilon \cup \{Q\} \Longrightarrow P \text{ iff } \Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow Q \to P$$

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T-79 5101 / Spring 2008 ML-5 Completeness **Theorem.** If  $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ , then  $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ . **Proof.** Let  $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$  hold. • By compactness of  $\models$ : there are finite sets  $\Sigma' \subseteq \Sigma$  and  $\Upsilon' = \{\phi_1, \dots, \phi_n\} \subseteq \Upsilon$  such that  $\Sigma' \models_{\mathbf{K}} \Upsilon' \Longrightarrow P$  holds.. • By the local deduction theorem for  $\models$ :  $\Sigma' \models_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \to (\phi_2 \to \cdots \to (\phi_n \to P) \cdots).$ • By the completeness of K-derivations:  $\Sigma' \vdash_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \to (\phi_2 \cdots \to (\phi_n \to P) \cdots)$ • By the local deduction theorem for  $\vdash$ :  $\Sigma' \vdash_{\mathbf{K}} \Upsilon' \Longrightarrow P.$ • By monotonicity of ⊢:  $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P.$ 

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# Soundness

**Theorem.** If  $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$  holds, then  $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P$  holds.

**Proof.** Let  $\Sigma \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$  hold.

- By compactness of  $\vdash$ : there are finite sets  $\Sigma' \subseteq \Sigma$  and  $\Upsilon' = \{\phi_1, \dots, \phi_n\} \subseteq \Upsilon$  such that  $\Sigma' \vdash_{\mathbf{K}} \Upsilon' \Longrightarrow P$ .
- By the local deduction theorem for  $\vdash$ :  $\Sigma' \vdash_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \to (\phi_2 \to \cdots \to (\phi_n \to P) \cdots).$
- By the soundness of **K**-derivations:  $\Sigma' \models_{\mathbf{K}} \emptyset \Longrightarrow \phi_1 \to (\phi_2 \cdots \to (\phi_n \to P) \cdots).$
- By the local deduction theorem for  $\models$ :  $\Sigma' \models_{\mathbf{K}} \Upsilon' \Longrightarrow P.$
- By the monotonicity of  $\models$ :  $\Sigma \models_{\mathbf{K}} \Upsilon \Longrightarrow P.$ 
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## 5. Examples of Hilbert-style Proof Systems

- Using the proof system for **K** and formulas characterizing properties of frames we can construct Hilbert-style proof systems for other frame logics.
- As the first example we consider the modal logic **T** where the frames are reflexive.
- The characteristic formula for reflexive frames: T:  $\Box P \rightarrow P$ .

**Proposition**.  $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$  iff  $\Sigma \cup \llbracket \mathbf{T} \rrbracket \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ 

 $\implies$  (Soundness and completeness of **K**-derivations)

**Proposition.**  $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$  iff  $\Sigma \cup \llbracket T \rrbracket \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ .

## Modal Logic T

Hence, a sound and complete Hilbert-style proof system for the modal logic **T** is obtained as follows:

Classical axioms: All tautologies

Modal axioms: All formulas of the form

 $\mathsf{K}:\ \Box(P\to Q)\to (\Box P\to \Box Q)$ 

T:  $\Box P \rightarrow P$ 

#### Modus Ponens -rule

N-rule

 $\implies$ 

```
Proposition. \Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P iff \Sigma \vdash_{\mathbf{T}} \Upsilon \Longrightarrow P.
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#### Modal Logic S5

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In a similar way for the frame logic S5 (equivalence frames):

**Proposition.**  $\Sigma \models_{\mathbf{S5}} \Upsilon \Longrightarrow P$  iff  $\Sigma \cup [\![T]\!] \cup [\![4]\!] \cup [\![B]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$  iff  $\Sigma \cup [\![T]\!] \cup [\![4]\!] \cup [\![5]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$  iff  $\Sigma \cup [\![T]\!] \cup [\![5]\!] \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ .

A Hilbert-style proof system for the modal logic  ${\bf S5}$  (modal axioms need to extended):

Modal axioms: All formulas of the form

K:  $\Box(P \to Q) \to (\Box P \to \Box Q)$ T:  $\Box P \to P$ 4:  $\Box P \to \Box \Box P$ 5:  $\neg \Box P \to \Box \neg \Box P$ 

**Proposition**.  $\Sigma \models_{\mathbf{S5}} \Upsilon \Longrightarrow P$  iff  $\Sigma \vdash_{\mathbf{S5}} \Upsilon \Longrightarrow P$ .

## Modal Logic KD45

KD45 is the collection of serial, transitive and eudlidian frames.

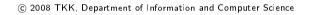
**Proposition.**  $\Sigma \models_{\mathbf{KD45}} \Upsilon \Longrightarrow P$  iff  $\Sigma \cup \llbracket D \rrbracket \cup \llbracket 4 \rrbracket \cup \llbracket 5 \rrbracket \vdash_{\mathbf{K}} \Upsilon \Longrightarrow P$ .

A Hilbert-style proof system for **KD45**:

**Modal axioms:** All formulas of the form

K: 
$$\Box(P \to Q) \to (\Box P \to \Box Q)$$
  
D:  $\Box P \to \Diamond P$   
4:  $\Box P \to \Box \Box P$   
5:  $\neg \Box P \to \Box \neg \Box P$ 

**Proposition**.  $\Sigma \models_{\mathbf{KD45}} \Upsilon \Longrightarrow P$  iff  $\Sigma \vdash_{\mathbf{KD45}} \Upsilon \Longrightarrow P$ .



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# Summary

- A proof system of a logic is a syntactic calculus for showing that a formula is valid/a logical consequence from a set of formulas in the logic.
- For modal logics Hilbert-style axiomatic proof systems are common in the literature although they do not lend themselves well to automation.
- The two most important properties of a proof system are soundness and completeness.
- Typically soundness is quite straightforward to establish.
- For many frame logics completeness of Hilbert-style systems can be shown using the canonical model construction which is here demonstrated for the modal logic **K**.
- Using formulas characterizing properties of frames it is straightforward to construct Hilbert-style proof systems for many other frame logics.