EXAMPLE MODAL LOGICS

1. Frame logics
2. Modal logics $\mathbf{K}$ and $\mathbf{T}$
3. Properties of frames
4. More examples of logics ($\mathbf{K4}$, $\mathbf{S4}$, $\mathbf{KB}$, $\mathbf{B}$, $\mathbf{S5}$, $\mathbf{D}$, $\mathbf{D4}$, and $\mathbf{DB}$)
5. Logics of belief
6. Deduction theorem and compactness

M. Fitting: *Basic Modal Logic*, 1.5 – 1.6 (pp. 384 – 387).

1. Frame Logics

The most well-known and frequently used modal logics are frame logics such that the set of valid formulas can be characterized by giving a collection $\mathbf{L}$ of frames $\langle S, R\rangle$ where the relation R satisfies chosen properties. We consider examples of such logics.

Example. $\mathbf{L}$ could be the collection of reflexive frames $\langle S, R\rangle$ where R is reflexive ($\forall x R(x,x)$ holds).

We have already shown that the set of $\mathbf{L}$-valid formulas is a normal propositional modal logic $\mathbf{L}$ that

1. includes all tautologies;
2. includes Q whenever it includes P and $P \rightarrow Q$;
3. is closed under substitution;
4. includes all formulas of the form $\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$;
5. includes $\Box P$ whenever it includes P.

Substitution Instances

Definition. If Σ is a set of formulas, $[\![\Sigma]\!]$ is the set of all substitution instances of the members of Σ.

- For example, if $\Sigma = \{P \rightarrow P\}$, then $[\![\Sigma]\!]$ contains, e.g., the formulas $P \rightarrow P$, $\quad \neg P \rightarrow \neg P$, $\quad \Box\Box Q \rightarrow \Box\Box Q$ $\quad$ and $(\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)) \rightarrow (\Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q))$.
- Sometimes we give names to formulas, for example,
$$\mathrm{I}\colon\ P \rightarrow P$$
- Then the set of substitution instances of the formula I is denoted by $[\![\mathrm{I}]\!]$, i.e., this is the set of formulas $[\![\{P \rightarrow P\}]\!]$.

2. Modal Logics $\mathbf{K}$ and $\mathbf{T}$

- Let $\mathbf{K}$ be the collection of all frames.
- Frame logic $\mathbf{K}$ is the weakest normal modal logic: if a formula is $\mathbf{K}$-valid, it is $\mathbf{L}$-valid in every normal modal logic $\mathbf{L}$.
- A characteristic formula
$$\mathrm{K}\colon\ \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$$

Proposition. Every formula in $[\![\mathrm{K}]\!]$ is $\mathbf{K}$-valid.

Proof. The proposition follows directly from Case 2 in the basic theorem of the possible world semantics. ■

Modal Logic T

Let $\mathbf{T}$ be the collection of all reflexive frames.

- For example, if $\Box$ is read as knowledge, reflexivity of the frames is reasonable: If the agent knows that P, then P is true.
  - Let $\langle S, R, v\rangle, s \Vdash \Box P$.
  - To guarantee that $\langle S, R, v\rangle, s \Vdash P$ holds it is sufficient that R is reflexive:

    If $\langle S, R, v\rangle, s \Vdash \Box P$ holds, then for every $t \in S$, such that sRt, $\langle S, R, v\rangle, t \Vdash P$ holds.

    When R is reflexive, sRs and $\langle S, R, v\rangle, s \Vdash P$ holds.

Modal Logic T

A characteristic formula for modal logic $\mathbf{T}$

$$\text{T: } \Box P \rightarrow P$$

is valid in a frame $\langle S, R\rangle$ iff R is reflexive (as we showed in Lecture ML-03).

$\Longrightarrow \quad \mathbf{T} = \mathbf{K} + [\![\mathrm{T}]\!]$

Proposition. $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{T}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Proof. ($\Longleftarrow$) Let $\Sigma \cup [\![\mathrm{T}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ hold.

Because $\mathbf{T} \subseteq \mathbf{K}$, then also $\Sigma \cup [\![\mathrm{T}]\!] \models_{\mathbf{T}} \Upsilon \Longrightarrow P$ holds.

Every member of $[\![\mathrm{T}]\!]$ is $\mathbf{T}$-valid as $\mathbf{T}$ is the collection of reflexive frames (See ML-03).

Hence, $\Sigma \models_{\mathbf{T}} \Upsilon \Longrightarrow P$ holds.

Proof (cont'd)

($\Longrightarrow$) Assume $\Sigma \cup [\![\mathrm{T}]\!] \not\models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ based on a frame $\langle S, R\rangle$ such that all formulas in $\Sigma \cup [\![\mathrm{T}]\!]$ are valid in the model and there is a world s in the model where $\langle S, R, v\rangle, s \Vdash \Upsilon \cup \{\neg P\}$ holds.

Let $R^* = R \cup \{(s,s) \mid s \in S\}$. We show that for every formula U for every world $s \in S$: $\langle S, R, v\rangle, s \Vdash U$ iff $\langle S, R^*, v\rangle, s \Vdash U$ by induction on the structure of the formula U:

- U is an atomic proposition Q: $\langle S, R, v\rangle, s \Vdash Q$ iff $\langle S, R^*, v\rangle, s \Vdash Q$.
- U is of the form $\neg Q$:

  $\langle S, R, v\rangle, s \Vdash \neg Q$ iff $\langle S, R, v\rangle, s \nVdash Q$ iff (by the inductive hypothesis) $\langle S, R^*, v\rangle, s \nVdash Q$ iff $\langle S, R^*, v\rangle, s \Vdash \neg Q$.

- U is of the form $Q \rightarrow Q'$ (can be shown as the case $\neg Q$).
- U is of the form $\Box Q$:

  ($\Leftarrow$) If $\langle S, R, v\rangle, s \nVdash \Box Q$ holds, there is a world t such that sRt and $\langle S, R, v\rangle, t \nVdash Q$. By the inductive hypothesis $\langle S, R^*, v\rangle, t \nVdash Q$. Now sR^*t and $\langle S, R^*, v\rangle, s \nVdash \Box Q$ holds.

  ($\Rightarrow$) If $\langle S, R^*, v\rangle, s \nVdash \Box Q$, then there is a world t such that sR^*t and $\langle S, R^*, v\rangle, t \nVdash Q$.

  1. If $t \neq s$, then sRt and $\langle S, R, v\rangle, s \nVdash \Box Q$.
  2. If $t = s$, then $\langle S, R^*, v\rangle, s \nVdash Q$ and $\langle S, R, v\rangle, s \nVdash Q$ by the inductive hypothesis.

     As $\Box Q \rightarrow Q$ is valid in the model $\langle S, R, v\rangle$, $\langle S, R, v\rangle, s \nVdash \Box Q$ holds.

Hence, $\langle S, R^*, v\rangle \models \Sigma$ and $\langle S, R^*, v\rangle, s \Vdash \Upsilon \cup \{\neg P\}$.

Hence, $\Sigma \not\models_{\mathbf{T}} \Upsilon \Longrightarrow P$ holds, since $\langle S, R^*\rangle$ is a reflexive frame. ■

3. Properties of Frames

Some properties of frames and corresponding modal formulas:

1. Reflexive:

$\forall s(sRs)$ $\quad\Box A \rightarrow A$

2. Symmetric:

$\forall s \forall t(sRt \rightarrow tRs)$ $\quad A \rightarrow \Box\Diamond A$

3. Serial:

$\forall s \exists t(sRt)$ $\quad\Box A \rightarrow \Diamond A$

4. Transitive:

$\forall s \forall t \forall u(sRt \wedge tRu \rightarrow sRu)$ $\quad\Box A \rightarrow \Box\Box A$

5. Euclidean:

$\forall s \forall t \forall u(sRt \wedge sRu \rightarrow tRu)$ $\quad\neg\Box A \rightarrow \Box\neg\Box A$

Properties of Frames—cont'd

6. Partially functional:

$\forall s \forall t \forall u(sRt \wedge sRu \rightarrow t = u)$ $\quad\Diamond A \rightarrow \Box A$

7. Functional:

$\forall s \exists! t(sRt)$ $\quad\Diamond A \leftrightarrow \Box A$

8. Weakly dense:

$\forall s \forall t(sRt \rightarrow \exists u(sRu \wedge uRt))$ $\quad\Box\Box A \rightarrow \Box A$

9. Weakly connected:

$\forall s \forall t \forall u(sRt \wedge sRu \rightarrow$ $\quad\Box(A \wedge \Box A \rightarrow B) \vee$

$tRu \vee t = u \vee uRt)$ $\quad\Box(B \wedge \Box B \rightarrow A)$

10. Weakly directed:

$\forall s \forall t \forall u(sRt \wedge sRu \rightarrow \exists v(tRv \wedge uRv))$ $\quad\Diamond\Box A \rightarrow \Box\Diamond A$

Characterizing Properties with Modal Formulas (I)

Theorem. Let $\mathcal{F} = \langle S, R\rangle$ be a frame. Then for each of the properties 1–10, if R satisfies the property, then every substitution instance of the corresponding formula is valid in $\mathcal{F}$.

Proof. 2. Let R be symmetric. We show that $\langle S, R\rangle \models [\![A \rightarrow \Box\Diamond A]\!]$.

Assume that there is a substitution instance $A \rightarrow \Box\Diamond A$, for which $\langle S, R\rangle \not\models A \rightarrow \Box\Diamond A$.
Then there is a model $\mathcal{M} = \langle S, R, v\rangle$ and a world $s \in S$ where $\mathcal{M}, s \Vdash A$ and $\mathcal{M}, s \nVdash \Box\Diamond A$. Hence, there is a world t such that sRt and $\mathcal{M}, t \nVdash \Diamond A$. Thus, for all t' such that tRt', $\mathcal{M}, t' \nVdash A$ holds. As R is symmetric, tRs and $\mathcal{M}, s \nVdash A$ holds, a contradiction. Hence, the assumption does not hold and $\langle S, R\rangle \models [\![A \rightarrow \Box\Diamond A]\!]$ holds.
Cases 3–10 can be proved in a similar way. ■

Characterizing Properties with Modal Formulas (II)

Theorem. If the substitution instances of one of the formulas 1–10 is valid in a frame $\mathcal{F} = \langle S, R\rangle$, then R satisfies the corresponding property.

Proof. 6. R partially functional vs. $\Diamond A \rightarrow \Box A$:

Let $\langle S, R\rangle \models [\![\Diamond A \rightarrow \Box A]\!]$ hold. Let us assume that R is not partially functional, i.e., $\forall s \forall t \forall u(sRt \wedge sRu \rightarrow t = u)$ does not hold.

Then there are $s, t, u \in S$ such that sRt, sRu but $t \neq u$.

Take v such that $v(t, P) = \text{true}$ and $v(u, P) = \text{false}$ for an atomic proposition P. Now $\langle S, R, v\rangle, s \Vdash \Diamond P$ and $\langle S, R, v\rangle, s \nVdash \Box P$ hold. Hence, $\langle S, R\rangle \models \Diamond P \rightarrow \Box P$ does not hold.

Thus, not all substitution instances of $\Diamond A \rightarrow \Box A$ are valid in the frame $\langle S, R\rangle$, a contradiction.
The rest of the cases can be shown in a similar way. ■

4. More Example Logics

Modal logic K4

- Let **K4** be the collection of transitive frames.
- A characteristic formula (positive introspection):

$$4 : \Box P \rightarrow \Box\Box P$$

Proposition. $\Sigma \models_{\mathbf{K4}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![4]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Modal logic S4

- Let **S4** be the collection of transitive and reflexive frames.

Proposition. $\Sigma \models_{\mathbf{S4}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![4]\!] \cup [\![\mathrm{T}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Example Logics—cont'd

Modal logic KB

- Let **KB** be the collection of symmetric frames.
- A characteristic formula

$$\mathrm{B} : P \rightarrow \Box\Diamond P$$

Proposition. $\Sigma \models_{\mathbf{KB}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{B}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Modal logic B

- Let **B** be the collection of symmetric and reflexive frames.

Proposition. $\Sigma \models_{\mathbf{B}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{B}]\!] \cup [\![\mathrm{T}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

$\Longrightarrow$ **KBT**

Modal Logic S5

- Let **S5** be the collection of equivalent frames (symmetric, reflexive and transitive).
- A characteristic formula (negative introspection):

$$5 : \neg\Box P \rightarrow \Box\neg\Box P$$

Proposition.

$\Sigma \models_{\mathbf{S5}} \Upsilon \Longrightarrow P$ iff
$\Sigma \cup [\![\mathrm{T}]\!] \cup [\![4]\!] \cup [\![\mathrm{B}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ iff
$\Sigma \cup [\![\mathrm{T}]\!] \cup [\![4]\!] \cup [\![5]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$ iff
$\Sigma \cup [\![\mathrm{T}]\!] \cup [\![5]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

$\Longrightarrow$ The logic of ideal knowledge and necessity.

Serial Modal Logic

Modal logic D

- Let **D** be the collection of serial (idealized) frames.
- A characteristic formula: $\mathrm{D} : \Box P \rightarrow \Diamond P$

Proposition. $\Sigma \models_{\mathbf{D}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{D}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Modal logic D4

- Let **D4** be the collection of serial and transitive frames.

Proposition. $\Sigma \models_{\mathbf{D4}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{D}]\!] \cup [\![4]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Modal logic DB

- Let **DB** be the collection of serial and symmetric frames.

Proposition. $\Sigma \models_{\mathbf{DB}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{D}]\!] \cup [\![\mathrm{B}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

5. Logics of Belief

- What is believed might not be true and, hence, in a logic of beliefs the frames are not necessarily reflexive.
- If we adopt positive and negative introspection, then we obtain modal logic **K45**.

  But $\neg\Box\bot$ is not **K45**-valid: $\langle\{s\},\emptyset,v\rangle,s \not\Vdash \Box\bot$.
- If we also assume serial frames, then we arrive at modal logic **KD45** (serial, transitive and euclidean frames).

Remark. Transitivity is not redundant: $\Box P \rightarrow \Box\Box P$ is not valid in serial and euclidean frames (**KD5**-valid).

Logics of Belief

- Formula $\neg\Box\bot$ is **KD45**-valid (since the frames are serial).
- Formula $\Box P \rightarrow P$ is not **KD45**-valid.
- Formula $\Box(\Box P \rightarrow P)$ is **KD45**-valid.

Proof. Let $\langle S,R\rangle$ be a **KD45**-frame.

Let $s \in S$ and sRt (such a world $t \in S$ always exists in a **KD45**-frame).

As the frame is euclidean: tRt holds (as sRt and sRt).

Hence, for all t such that sRt holds, also tRt holds.

Hence, $\langle S,R,v\rangle,s \Vdash \Box(\Box P \rightarrow P)$,

because for every t such that sRt, $\langle S,R,v\rangle,t \Vdash \Box P \rightarrow P$. ■

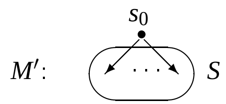
Simpler Collections of Frames for S5 and KD45

- For modal logic **S5** it is enough to consider only universal frames, i.e., frames $\langle S,R\rangle$ where $R = \{\langle s,t\rangle \mid s,t \in S\}$.

Proposition. If a formula P is true in a model bases on a **S5**-frame, then P is true in a model based on a universal frame.

Proposition. If a formula P is true in a model based on a **KD45**-frame, then P is true in a model M' of the form

$$M' = \langle\{s_0\} \cup S, \{\langle s,t\rangle \mid s \in \{s_0\} \cup S, t \in S\}, v\rangle.$$



6. Deduction Theorem and Compactness

- For all logics considered above the global deduction theorem holds:

  $\Sigma \cup \{Q\} \models_{\mathbf{L}} \Upsilon \Longrightarrow P$ iff

  for some n it holds that $\Sigma \models_{\mathbf{L}} \Upsilon \cup \{\Box^0 Q, \Box^1 Q, \ldots, \Box^n Q\} \Longrightarrow P$.
- In addition the logics are compact:

  If $\Sigma \models_{\mathbf{L}} \Upsilon \Longrightarrow P$, then there are finite subsets $\Sigma_0 \subseteq \Sigma$ and $\Upsilon_0 \subseteq \Upsilon$ such that $\Sigma_0 \models_{\mathbf{L}} \Upsilon_0 \Longrightarrow P$.
- However, not all modal logics (or even frame logics have these properties.

Modal Logic GL

- Let **GL** be the collection of transitive, irreflexive and finite frames (or the collection of transitive frames where there is no infinite sequence of worlds with each accessible from its predecessor.
- This does not correspond to any formula in (first-order) predicate logic expressing the properties of the frame.
- A characteristic formula

$$\mathrm{GL} : \Box(\Box P \to P) \to \Box P$$

- Global deduction theorem does not hold and **GL** is not compact.

Proposition. If Σ and Υ are finite sets of formulas, then $\Sigma \models_{\mathbf{GL}} \Upsilon \Longrightarrow P$ iff $\Sigma \cup [\![\mathrm{GL}]\!] \models_{\mathbf{K}} \Upsilon \Longrightarrow P$.

Summary

- The most well-known and frequently used modal logics are all frame logics.
- Interesting properties of frames can be expressed using characterizing modal formulas.
- Logical consequence in many frame logics can be captured by adding the characterizing modal formulas for the properties of the frames as global premises.
- This leads to natural Hilbert-style proof systems for these logics (as will be shown in the next lecture).