Basic Properties of Possible World Models

1. Relationship to predicate logic
2. Generated submodels
3. p-morphisms of frames
4. Logical consequence
5. Deduction theorem

M. Fitting: *Basic Modal Logic*, 1.3 – 1.4 (p. 377 – 384).

1. Relationship to Predicate Logic

- A frame can be seen as a structure in predicate logic which has one two-argument predicate symbol (R):

  A predicate logic structure $\mathcal{S}$ can be given in the form $\langle U, R^{\mathcal{S}} \rangle$, where U is the universum of the structure and relation $R^{\mathcal{S}} \subseteq U \times U$ gives the interpretation of the predicate symbol R.

- Propositional modal logic can be used as a replacement of quantified logic to characterize classes of structures in classical logic in the following sense:

  for a formula P in predicate logic a corresponding propositional modal formula P' can be given such that P is true in a structure iff P' is valid in the structure (seen as a frame in modal logic).

Characterizing Reflexive Relations

Example. Formula $\Box P \to P$ is valid in a frame $\langle S, R \rangle$ iff R is reflexive (i.e. $\forall x R(x,x)$ is true in the structure $\langle S, R \rangle$).

Proof. ($\Longleftarrow$) Let $R \subseteq S \times S$ reflexive relation, $\mathcal{M} = \langle S, R, v \rangle$ a model based on the frame $\langle S, R \rangle$ and $s \in S$.

If $\mathcal{M}, s \Vdash \Box P$, then $\mathcal{M}, s \Vdash P$, as sRs. Hence, $\mathcal{M}, s \Vdash \Box P \to P$.

Thus $\Box P \to P$ is valid in the frame $\langle S, R \rangle$.

($\Longrightarrow$) Assume that R is not reflexive in the frame $\langle S, R \rangle$. Then there is a world $s_0 \in S$ such that $s_0 R s_0$ does not hold.

Let $v(s, P) = \text{true}$ if $s_0 R s$ otherwise $v(s, P) = \text{false}$. Now $v(s_0, P) = \text{false}$.

Let $\mathcal{M} = \langle S, R, v \rangle$. Then $\mathcal{M}, s_0 \Vdash \Box P$ but $\mathcal{M}, s_0 \not\Vdash P$.

Thus, $\mathcal{M}, s_0 \not\Vdash \Box P \to P$ and so $\Box P \to P$ is not valid in the frame $\langle S, R \rangle$. ■

Definability

- As shown above the formula $\forall x R(x,x)$ in predicate logic can be expressed as a propositional modal formula $\Box P \to P$.
- How far can this be pushed?

  What kind of formulas in predicate logic cannot be expressed as propositional modal formulas?
- Next we consider three operations on frames that preserve validity (generated submodels, disjoint union and p-morphism) and can be used to establish some limits for the expressivity of modal logic.

  ☞ For example, formulas $\exists x \exists y R(x,y)$, $\forall x \forall y R(x,y)$ and $\forall x \neg R(x,x)$ cannot be expressed as propositional modal formulas.

2. Generated Submodels

Definition. Let $\langle S, R\rangle$ be a frame and $\emptyset \subset S_0 \subseteq S$ a set of worlds. The subframe generated by S_0 is the frame $\langle S^*, R^*\rangle$, where S^* is the smallest R-closed subset of S that contains S_0 and R^* is R restricted to the set S^* ($R^* = \{\langle s_1, s_2\rangle \in R \mid s_1, s_2 \in S^*\} = R \cap (S^* \times S^*)$).

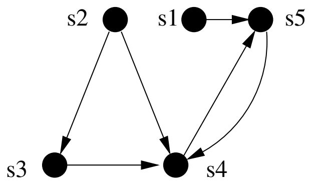
A subset H of S is R-closed if $t \in H$ whenever $s \in H$ and sRt.

Remark. S^* is the set of worlds that can be reached from S_0 (transitively).

Definition. A generated submodel of $\langle S, R, v\rangle$ is a model $\langle S^*, R^*, v^*\rangle$, where $\langle S^*, R^*\rangle$ is a generated subframe and v^* is v restricted to S^*. A subframe is generated if it is generated by some set of worlds.

Example

Consider the following frame:



- The set $S_0 = \{s2\}$ generates a subframe where $S^* = \{s2, s3, s4, s5\}$ and $R^* = R \cap (S^* \times S^*)$ gives the edge relation between the worlds.
- The set $S_0 = \{s5\}$ generates a subframe where $S^* = \{s4, s5\}$.
- The set $S_0 = \{s1, s3\}$ generates a subframe where $S^* = \{s1, s3, s4, s5\}$.

Properties of Generated Submodels

Proposition. If $\mathcal{M}^*$ is a generated submodel of $\mathcal{M}$, then for every formula P and world s in $\mathcal{M}^*$ it holds that

$$\mathcal{M}, s \Vdash P \text{ iff } \mathcal{M}^*, s \Vdash P.$$

Example. Let $S = \{s, t_1, t_2\}$ and $R = \{\langle t_1, t_2\rangle\}$.

The subframe generated by the set $\{s\}$ is $\langle S^*, R^*\rangle$ where $S^* = \{s\}$ and $R^* = \{\}$.

Now $\exists x \exists y R(x, y)$ is true in the frame $\langle S, R\rangle$ and false in the frame $\langle S^*, R^*\rangle$.

But for every modal formula P and every valuation v it holds that $\langle S, R, v\rangle, s \Vdash P$ iff $\langle S^*, R^*, v^*\rangle, s \Vdash P$ by the proposition above.

$\Longrightarrow$ There is no propositional modal formula P such that P is valid in a frame $\langle S, R\rangle$ iff R satisfies $\exists x \exists y R(x, y)$.

Disjoint Union of Frames

Disjoint union of frames: take copies of the frames such that their sets of worlds are disjoint and then form the union of the worlds and the accessibility relations.

Proposition. A modal formula P is valid in the disjoint union of two frames iff P is valid in both of the frames.

Example. Let $S = \{s\}$ and $R = \{\langle s, s\rangle\}$.

The disjoint union of the frame $\langle S, R\rangle$ with itself is the frame $\langle S, R\rangle \dot\cup \langle S, R\rangle = \langle S', R'\rangle$ where $S' = \{s_1, s_2\}$ and $R' = \{\langle s_1, s_1\rangle, \langle s_2, s_2\rangle\}$.

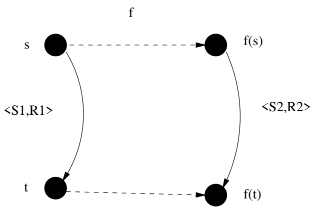
Now $\forall x \forall y R(x, y)$ is true in the frame $\langle S, R\rangle$ and false in the frame $\langle S', R'\rangle$ but for all modal formulas P it holds that $\langle S, R\rangle \models P$ iff $\langle S', R'\rangle \models P$ by the proposition above.

$\Longrightarrow$ There is no propositional modal formula P such that P is valid in a frame $\langle S, R\rangle$ iff R satisfies $\forall x \forall y R(x, y)$.

3. Frame p-morphism

Definition. Mapping $f : \langle S_1, R_1 \rangle \mapsto \langle S_2, R_2 \rangle$ is a p-morphism iff

1. f maps the set S_1 onto the set S_2 surjectively such that for every $s \in S_2$ there is $t \in S_1$ with $s = f(t)$,
2. for every $s, t \in S_1$, if sR_1t, then $f(s)R_2f(t)$ and
3. for every $s \in S_1$ and $u \in S_2$, if $f(s)R_2u$, then there is $t \in S_1$ such that sR_1t and $f(t) = u$.

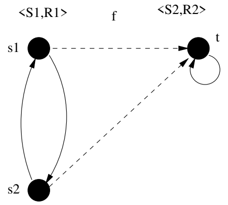


Example

Let $\langle S_1, R_1 \rangle$ be a frame where $S_1 = \{s_1, s_2\}$ and $R_1 = \{\langle s_1, s_2 \rangle, \langle s_2, s_1 \rangle\}$.

and let $\langle S_2, R_2 \rangle$ be a frame where $S_2 = \{t\}$ and $R_2 = \{\langle t, t \rangle\}$.

Consider a mapping f such that $f(s_1) = f(s_2) = t$.



☞ Function f is a p-morphism.

Properties of Frame p-morphism

Proposition. If there is a p-morphism from a frame $\langle S_1, R_1 \rangle$ to a frame $\langle S_2, R_2 \rangle$, then every formula valid in the first frame is valid in the second frame.

Example. (Continued)

Now $\forall x \neg R(x,x)$ is true in $\langle S_1, R_1 \rangle$ and false in $\langle S_2, R_2 \rangle$.

However, for all modal formulas P: if P valid in the frame $\langle S_1, R_1 \rangle$, then P is valid in the frame $\langle S_2, R_2 \rangle$ by the proposition above.

$\Longrightarrow$ There is no propositional modal formula P such that P is valid in a frame $\langle S, R \rangle$ iff R satisfies $\forall x \neg R(x,x)$.

4. Logical Consequence

- In propositional logic logical consequence ($\Sigma \models P$) requires that

  P is true in every valuation where the premises Σ are true.

- In modal logic there seems to be two ways of interpreting premises:
  - As "logical truths":

    For example, if the agent knows that P then P is true ($\Box P \rightarrow P$) from which it seems reasonable to conclude that this is known ($\Box(\Box P \rightarrow P)$).
  - As "factual truths":

    For example, that it rains today in Bombay (Q) for which it seems not to be reasonable to conclude that this is known ($\Box Q$).

Logical Consequence

- Hence, in modal logic there are many possible ways to interpret logical consequence depending on how the premises are taken:
  - For all models it holds that P is true in every possible world where Σ is true.
  - P is valid in every model where Σ is valid.
  - P is valid in every frame where Σ is valid.
- Next we introduce a notion of logic consequence that combines all the three interpretations.

Defining Logical Consequence

Definition. Let $\mathbf{L}$ be a collection of frames, Σ and Υ sets of modal formulas and P a modal formula. Formula P is a logical consequence of the global premises Σ and local premises Υ in the frame logic $\mathbf{L}$ ($\Sigma \models_{\mathbf{L}} \Upsilon \Longrightarrow P$) iff

for all frames $\langle S,R\rangle \in \mathbf{L}$, for all models $\mathcal{M} = \langle S,R,v\rangle$ based on $\langle S,R\rangle$ where $\mathcal{M} \models \Sigma$, for every possible world $s \in S$ where $\mathcal{M},s \models \Upsilon$ also $\mathcal{M},s \models P$ holds.

Example. Let $\mathbf{L}$ the collection of all frames.

If $\Sigma \models_{\mathbf{L}} \emptyset \Longrightarrow P$, then $\Sigma \models_{\mathbf{L}} \emptyset \Longrightarrow \Box P$.

Example

- Notice that in order to establish that logical consequence does not hold (denoted by $\Sigma \not\models_{\mathbf{L}} \Upsilon \Longrightarrow P$) it is enough to give
  - a frame $\langle S,R\rangle \in \mathbf{L}$,
  - a model $\mathcal{M} = \langle S,R,v\rangle$ based on $\langle S,R\rangle$ where $\mathcal{M} \models \Sigma$ holds, and
  - a world $s \in S$ where $\mathcal{M},s \models \Upsilon$ holds but $\mathcal{M},s \models P$ does not hold.
- Note that $\{P\} \models_{\mathbf{L}} \emptyset \Longrightarrow \Box P$ holds.
- However, $\emptyset \models_{\mathbf{L}} \{P\} \Longrightarrow \Box P$ does not when $\mathbf{L}$ is the collection of all frames.

  Proof: Consider a frame $\langle S,R\rangle$ where $S = \{s,t\}$ and $R = \{(s,t)\}$ and a model based on this $\mathcal{M} = \langle S,R,v\rangle$ where $v(s,P) = \text{true}$, $v(t,P) = \text{false}$.

  Now in the world $s \in S$: $\mathcal{M},s \models \{P\}$ but $\mathcal{M},s \not\models \Box P$.

  Hence, $\emptyset \not\models_{\mathbf{L}} \{P\} \Longrightarrow \Box P$ holds.

Properties of the Logical Consequence Relation

Definition. If $\mathbf{L}_1$ and $\mathbf{L}_2$ are two collections of frames and $\mathbf{L}_2 \subseteq \mathbf{L}_1$, we say that $\mathbf{L}_1$ is weaker than $\mathbf{L}_2$ and write $\mathbf{L}_1 \leq \mathbf{L}_2$.

Note that if $\mathbf{L}_1$ weaker than $\mathbf{L}_2$, then $\mathbf{L}_1$-valid formulas are also $\mathbf{L}_2$-valid.

Theorem. (Monotonicity) Let $\Sigma_1 \subseteq \Sigma_2$, $\Upsilon_1 \subseteq \Upsilon_2$ and $\mathbf{L}_1 \leq \mathbf{L}_2$. Then if $\Sigma_1 \models_{\mathbf{L}_1} \Upsilon_1 \Longrightarrow P$, then $\Sigma_2 \models_{\mathbf{L}_2} \Upsilon_2 \Longrightarrow P$.

Theorem. (Replacement) Let Q and Q' be alike formulas except that at some places where Q contains P as a subformula, Q' has P' as the subformula. Then

$\Sigma \cup \{P \leftrightarrow P'\} \models_{\mathbf{L}} \emptyset \Longrightarrow Q \leftrightarrow Q'$.

Example. By the theorem $\{P \leftrightarrow P'\} \models_{\mathbf{L}} \{\} \Longrightarrow \Box P \leftrightarrow \Box P'$ holds.

However, $\{\} \models_{\mathbf{L}} \{P \leftrightarrow P'\} \Longrightarrow \Box P \leftrightarrow \Box P'$ does not hold.

4. Deduction Theorem

In propositional logic: $\Sigma \cup \{Q\} \models P$ iff $\Sigma \models Q \rightarrow P$.

Theorem. (Local Deduction Theorem)
$\Sigma \models_{\mathbf{L}} \Upsilon \cup \{Q\} \Longrightarrow P$ iff $\Sigma \models_{\mathbf{L}} \Upsilon \Longrightarrow Q \rightarrow P$.

Definition. For a modal formula P, $\Box^0 P = P$ and $\Box^n P = \Box(\Box^{n-1} P)$.

Example. $\Box^3 P = \Box\Box\Box P$.

Global Deduction Theorem:

$\Sigma \cup \{Q\} \models_{\mathbf{L}} \Upsilon \Longrightarrow P$ iff for some n
$\Sigma \models_{\mathbf{L}} \Upsilon \cup \{\Box^0 Q, \Box^1 Q, \ldots, \Box^n Q\} \Longrightarrow P$.

(This is not valid for all collections of frames $\mathbf{L}$.)

Global Deduction Theorem and Compactness

Theorem. (Global Deduction Theorem I)
If for some n, $\Sigma \models_{\mathbf{L}} \Upsilon \cup \{\Box^0 Q, \Box^1 Q, \ldots, \Box^n Q\} \Longrightarrow P$, then
$\Sigma \cup \{Q\} \models_{\mathbf{L}} \Upsilon \Longrightarrow P$.

Theorem. (Global Deduction Theorem II)
Let $\mathbf{L}$ be a collection of frames that is closed under ultraproducts.
Then if $\Sigma \cup \{Q\} \models_{\mathbf{L}} \Upsilon \Longrightarrow P$, then
$\Sigma \models_{\mathbf{L}} \Upsilon \cup \{\Box^0 Q, \Box^1 Q, \ldots, \Box^n Q\} \Longrightarrow P$ for some n.

Theorem. (Compactness) Let $\mathbf{L}$ be a collection of frames that is closed under ultraproducts. If $\Sigma \models_{\mathbf{L}} \Upsilon \Longrightarrow P$, then there are finite sets $\Sigma_0 \subseteq \Sigma$ and $\Upsilon_0 \subseteq \Upsilon$ such that $\Sigma_0 \models_{\mathbf{L}} \Upsilon_0 \Longrightarrow P$.

(In this course we do not consider the ultraproducts mentioned above but give examples of logics where the condition is satisfied.)

Summary

- Frames in modal logic can be seen as simple structures in predicate logic giving truth values for formulas in predicate logic with one two-argument predicate symbol R.
- Modal logic formulas can be used to characterize classes of structures in predicate logic.
- However, validity preserving operations on frames (generated submodels, disjoint union and p-morphism) show that the expressivity of modal logic is limited.
- Logical consequence in modal logic is more complicated than in classical logic: given a logic (a collection of frames) two classes of premises, global and local ones, need to be considered.
- The resulting logical consequence is monotonic and satisfies the deduction theorem for local premises. For most widely used logics the relation satisfies also the deduction theorem for global premises and remains compact.