PROPOSITIONAL MODAL LOGICS

1. Basic concepts and definitions
2. Possible world semantics
3. Basic properties of the possible world semantics

M. Fitting: *Basic Modal Logic*, Sections 1.1 – 1.2 (p. 372 – 377).

1. Basic Concepts and Definitions

- We consider first propositional modal logics which have one modal operator $\Box$ and its dual operator $\Diamond$ ($\neg\Box\neg$).
- The operator can be given many alternative interpretations:

$\Box P$	$\Diamond P$
necessarily P	possibly P
always P	
ought to be that P	
know that P	
believe that P	

Syntax of Modal Logic

Let $\Phi = \{p_1, p_2, \ldots\}$ be a set of (atomic) propositions.

The class of *propositional modal formulas* is defined as follows:

1. Every atomic proposition is a *formula*.
2. $\top$ and $\bot$ are *formulas*.
3. If P and Q are formulas, then $\neg P$, $(P \rightarrow Q)$, $\Box P$ are *formulas*.
4. There are no other formulas.

Note the usual abbreviations:

$$\Diamond P \equiv_{\text{def}} \neg\Box\neg P \qquad P \vee Q \equiv_{\text{def}} \neg P \rightarrow Q$$

$$P \wedge Q \equiv_{\text{def}} \neg(P \rightarrow \neg Q) \qquad P \leftrightarrow Q \equiv_{\text{def}} \neg((P \rightarrow Q) \rightarrow \neg(Q \rightarrow P))$$

Tautologies

Definition. If a formula Q results from formula P by replacing the atomic propositions of P uniformly by formulas, we say that Q results from P by substitution.

A formula that results by substitution from a tautology in propositional logic is called a classical tautology.

Example. Formula

$$(\Box\Box P \rightarrow (R \rightarrow \Box S)) \rightarrow (\Box\Box P \rightarrow (R \rightarrow \Box S))$$

is a classical tautology which results from a propositional tautology $P \rightarrow P$ by substituting proposition P by formula $(\Box\Box P \rightarrow (R \rightarrow \Box S))$.

Tautologies—cont'd

Classical tautologies can be identified using propositional tableaux as follows: Construct a semantical tableau for the negation of the formula by treating subformulas of the form $\Box P$ as atomic (no tableau rule is applied to them).

If the tableau closes, the formula is a classical tautology.

Example. Consider a formula $(\Box\neg\Box(P\to Q)\wedge\Box(\neg P\wedge\neg Q))\to(\Box\neg\Box(P\to Q)\vee\Box(\neg P\wedge\neg Q))$

As the tableau (on the right) closes, the formula is a classical tautology.

1.	$\neg((\Box\neg\Box(P\to Q)\wedge\Box(\neg P\wedge\neg Q))\to(\Box\neg\Box(P\to Q)\vee\Box(\neg P\wedge\neg Q)))$	
2.	$\Box\neg\Box(P\to Q)\wedge\Box(\neg P\wedge\neg Q)$	(1)
3.	$\neg(\Box\neg\Box(P\to Q)\vee\Box(\neg P\wedge\neg Q))$	(1)
4.	$\Box\neg\Box(P\to Q)$	(2)
5.	$\Box(\neg P\wedge\neg Q)$	(2)
6.	$\neg\Box\neg\Box(P\to Q)$	(3)
7.	$\neg\Box(\neg P\wedge\neg Q)$	(3)
	$\times$	

Logic = A Set of Valid Formulas

Definition. *Propositional modal logic $\mathcal{L}$* is a set of propositional modal formulas such that

1. all (classical) tautologies are in $\mathcal{L}$;
2. $\mathcal{L}$ is closed under modus ponens (if $P\in\mathcal{L}$ and $P\to Q\in\mathcal{L}$, then $Q\in\mathcal{L}$) ;
3. $\mathcal{L}$ closed under substitution: (if $P\in\mathcal{L}$ and Q results from P by substitution, then $Q\in\mathcal{L}$).

Example. (1) The set of classical tautologies is a propositional modal logic.

(2) The set of all propositional modal formulas is a propositional modal logic.

2. Possible World Semantics

Definition. A *frame* is a pair $\mathcal{F}=\langle S,R\rangle$, where S is a non-empty set and R is a binary relation on S ($R\subseteq S\times S$).

The members of S are called *possible worlds* and R is the accessibility relation between worlds: for worlds s_1 and s_2, if $(s_1,s_2)\in R$ (often denoted by s_1Rs_2), then s_2 is said to be *accessible* from s_1.

Definition. A *valuation* in a frame $\langle S,R\rangle$ is a function v mapping possible worlds and propositions to truth values: for all $s\in S$ and propositions P, $v(s,P)$ is either true or false.

(Alternatively: a valuation $v:S\to 2^{\Phi}$ is a function such that $v(s)$ is the set of propositions true in the world s.)

Definition. A *model* is a triple $\mathcal{M}=\langle S,R,v\rangle$ where $\langle S,R\rangle$ is a frame and v a valuation in this frame. A model $\mathcal{M}=\langle S,R,v\rangle$ *is based on the frame* $\langle S,R\rangle$.

Truth in a Model

Relation $\mathcal{M},s\Vdash P$ (often denoted also by $\mathcal{M},s\models P$), tells whether formula P is true in the possible world s of the model $\mathcal{M}$ and it is defined as follows:

Definition. Let $\mathcal{M}=\langle S,R,v\rangle$ be a model.

1. $\mathcal{M},s\Vdash P$ iff $v(s,P)=\text{true}$, when P is an atomic proposition.
2. $\mathcal{M},s\nVdash\bot$ and $\mathcal{M},s\Vdash\top$.
3. $\mathcal{M},s\Vdash\neg P$ iff $\mathcal{M},s\nVdash P$.
4. $\mathcal{M},s\Vdash P\to Q$ iff $\mathcal{M},s\nVdash P$ or $\mathcal{M},s\Vdash Q$.
5. $\mathcal{M},s\Vdash\Box P$ iff $\mathcal{M},t\Vdash P$ for every $t\in S$ such that sRt.

Note: $\mathcal{M},s\Vdash\Diamond P$ iff $\mathcal{M},t\Vdash P$ for some $t\in S$ such that sRt.

Truth in a Model—cont'd

Example. Let $\mathcal{M} = \langle S, R, v\rangle$ be a model where

$S = \{s,t\}$ and $R = \{\langle s,t\rangle\}$

$v(s,P) = \text{true}$ (or $v(s) = \{P\}$)

$v(t,P) = \text{false}$ (or $v(t) = \{\}$)

$$\mathcal{M}: \underset{s}{\{P\}} \longrightarrow \underset{t}{\{\}}$$

- $\mathcal{M}, s \nVdash \Box P$.
- $\mathcal{M}, t \Vdash \Box P$.
- $\mathcal{M}, s \Vdash \Diamond\neg P$.
- $\mathcal{M}, s \Vdash \Box P \to P$.
- $\mathcal{M}, t \nVdash \Box P \to P$.

Validity

Definition.

- Formula P is *true* in the world s of the model $\mathcal{M} = \langle S, R, v\rangle$ if $\mathcal{M}, s \Vdash P$.
- Formula P is *valid in the model* $\mathcal{M} = \langle S, R, v\rangle$, if P is true in every world s of the model $\mathcal{M} = \langle S, R, v\rangle$ ($\mathcal{M} \models P$).
- Formula P is *valid in a non-empty collection of models* $\mathbf{C}$ if P is valid in every model in $\mathbf{C}$ ($\mathbf{C} \models P$).
- Formula P is *valid in a non-empty collection of frames* $\mathbf{F}$ if P is valid in the collection of all models based on the frames in $\mathbf{F}$ ($\mathbf{F} \models P$).

Validity—cont'd

- If formula P is valid in a collection of models/frames $\mathbf{C}$, then it is said that P is $\mathbf{C}$-valid.
- A set of formulas Σ is said to be valid in a collection of models/frames $\mathbf{C}$ ($\mathbf{C} \models \Sigma$) when every formula in Σ is valid in the collection $\mathbf{C}$.

Example. Let $\mathcal{M} = \langle S, R, v\rangle$ where $S = \{s,t\}$, $R = \{\langle s,t\rangle\}$ and $v(s,P) = \text{true}$, $v(t,P) = \text{false}$.

- $\Box P \to P$ is not valid in $\mathcal{M}$ as $\mathcal{M}, t \nVdash \Box P \to P$.
- $P \vee \Box P$ is valid in $\mathcal{M}$.
- $\Box\Box P$ is $\mathbf{F}$-valid where $\mathbf{F} = \{\langle S, R\rangle\}$.

3. Basic Properties of Possible World Semantics

Theorem. (Basic theorem of the possible world semantics)
If $\mathbf{C}$ is a collection of models, the set of $\mathbf{C}$-valid formulas

1. includes all tautologies;
2. includes all formulas of the form $\Box(P \to Q) \to (\Box P \to \Box Q)$;
3. includes Q whenever it includes P and $P \to Q$;
4. includes $\Box P$ whenever it includes P;

and if $\mathbf{F}$ is a collection of frames, the set of $\mathbf{F}$-valid formulas is closed under substitution.

Proof. (1.) Consider an arbitrary classical tautology. It is true in every world of every model. Hence, it is **C**-valid.

(2.) Let $\mathcal{M} = \langle S, R, v \rangle$ be a model in **C** and s a world in S. Assume that there is a formula of the form $\Box(P \to Q) \to (\Box P \to \Box Q)$ which is not true in s. Then

(i) $\mathcal{M}, s \Vdash \Box(P \to Q)$ and

(ii) $\mathcal{M}, s \Vdash \Box P$ but

(iii) $\mathcal{M}, s \not\Vdash \Box Q$.

By (iii) there is a world $t \in S$ such that sRt and $\mathcal{M}, t \not\Vdash Q$. But then $\mathcal{M}, t \Vdash P \to Q$ and $\mathcal{M}, t \Vdash P$ by (i & ii). Hence, $\mathcal{M}, t \Vdash Q$, a contradiction.

Thus, every formula of the form $\Box(P \to Q) \to (\Box P \to \Box Q)$ is true in any world s and, hence, **C**-valid.

Proof. (cont'd).

(3.) Let P and $P \to Q$ be **C**-valid.

Let $\mathcal{M} = \langle S, R, v \rangle$ be a model in **C** and s a world in S.

Then P and $P \to Q$ are true in the world s and, thus, Q is true in s. Hence, Q is true in every world in each model in **C** which implies that Q is **C**-valid.

(4.) Let P be **C**-valid.

Let $\mathcal{M} = \langle S, R, v \rangle$ by a model in **C** and s a world in S.

The for each world $t \in S$ such that sRt, $\mathcal{M}, t \Vdash P$ and $\mathcal{M}, s \Vdash \Box P$. Thus, $\Box P$ is **C**-valid.

Proof. (cont'd).

Let **F** is a non-empty collection of frames. We show that the set of **F**-valid formulas is closed under substitution, that is,

if X is **F**-valid, then the formula $\sigma(X)$ is **F**-valid

where $\sigma(X)$ results from X by a substitution σ, in which each proposition P in X is replaced uniformly by the formula $\sigma(P)$.

We establish the result by showing that if $\sigma(X)$ is not **F**-valid, then X is not **F**-valid.

Assume $\sigma(X)$ is not **F**-valid, then there is a frame $\langle S, R \rangle \in \mathbf{F}$, valuation v and a world $t \in S$ such that $\mathcal{M}, t \not\Vdash \sigma(X)$ where $\mathcal{M} = \langle S, R, v \rangle$.

Proof. (cont'd).

Let $\mathcal{M}' = \langle S, R, v' \rangle$ where $v'(P, s) = \text{true}$ iff $\mathcal{M}, s \Vdash \sigma(P)$ for atomic propositions P.

We show by structural induction for every formula Z, that for each world $s \in S$ it holds that $\mathcal{M}', s \Vdash Z$ iff $\mathcal{M}, s \Vdash \sigma(Z)$.

- If Z is an atomic proposition, then $\mathcal{M}', s \Vdash Z$ iff $v'(Z, s) = \text{true}$ iff $\mathcal{M}, s \Vdash \sigma(Z)$.
- If Z is of the form $\top$ or $\bot$, then $\mathcal{M}', s \Vdash Z$ iff $\mathcal{M}, s \Vdash \sigma(Z)$.
- If Z is of the form $\neg Z'$, then $\mathcal{M}', s \Vdash \neg Z'$ iff $\mathcal{M}', s \not\Vdash Z'$. By the inductive hypothesis this hold exactly when $\mathcal{M}, s \not\Vdash \sigma(Z')$ which holds iff $\mathcal{M}, s \Vdash \neg\sigma(Z')[= \sigma(\neg Z')]$.

Proof. (cont'd).

- If Z is of the form $Z' \to Z''$, then $\mathcal{M}', s \mid\!\mid\!\!- Z' \to Z''$ iff $\mathcal{M}', s \mid\!\mid\!\!\not- Z'$ or $\mathcal{M}', s \mid\!\mid\!\!- Z''$ iff $\mathcal{M}, s \mid\!\mid\!\!\not- \sigma(Z')$ or $\mathcal{M}, s \mid\!\mid\!\!- \sigma(Z'')$ [IH] iff $\mathcal{M}, s \mid\!\mid\!\!- \sigma(Z') \to \sigma(Z'')[= \sigma(Z' \to Z'')]$.
- If Z is of the form $\Box Z'$, then $\mathcal{M}', s \mid\!\mid\!\!- \Box Z'$ iff for all $t \in S$ such that sRt it holds that $\mathcal{M}', t \mid\!\mid\!\!- Z'$ iff for all $t \in S$ such that sRt it holds that $\mathcal{M}, t \mid\!\mid\!\!- \sigma(Z')$ [IH] iff $\mathcal{M}, s \mid\!\mid\!\!- \Box\sigma(Z')[= \sigma(\Box Z')]$.

Hence, if $\mathcal{M}, t \mid\!\mid\!\!\not- \sigma(X)$, then $\mathcal{M}', t \mid\!\mid\!\!\not- X$. This implies that X is not $\mathbf{F}$-valid. ■

Normal Modal Logics

Corollary. If $\mathbf{L}$ is a non-empty set of frames, then the set of $\mathbf{L}$-valid formulas is a propositional modal logic.

From now on we use the notation $\mathbf{L}$ in a double way: $\mathbf{L}$ can refer to

(1) a collection of frames or
(2) the set of formulas that are valid in $\mathbf{L}$.

Definition. Propositional modal logic is called normal if it includes (i) all formulas of the form $\Box(P \to Q) \to (\Box P \to \Box Q)$ and (ii) the formula $\Box P$ whenever P is included.

Definition. A set of formulas $\mathcal{L}$ is called a *frame logic* is $\mathcal{L}$ is the set of $\mathbf{L}$-valid formulas for some non-empty collection of frames $\mathbf{L}$.

We use $\mathbf{L}$ to denote a logic and the corresponding collection of frames.

Summary

- The language of propositional modal formulas with one modal operator ($\Box$) is introduced and the notion of classical tautologies is defined.
- Different modal logics in this language are identified by the valid formulas in the logic.
- The possible world semantics for modal formulas is introduced and key concepts are presented: truth in a model and validity.
- Some basic properties of the possible world semantics are proved. The notion of structural induction is illustrated through an example use of it in one of the proofs.
- The notion of normal modal logics is introduced.