

## Parallel and Distributed Systems

## Tutorial 4 – Solutions

Formally, the automata  $\mathcal{A}_1$ ,  $\mathcal{A}_2$ ,  $\mathcal{A}_3$  have the definitions

$$\boxed{\mathcal{A}_1 = (\Sigma_1, S_1, S_1^0, \Delta_1, F_1):}$$

$$\Sigma_1 = \{a, b\},$$

$$S_1 = \{q_0, q_1, q_2\},$$

$$S_1^0 = \{q_0\},$$

$$\Delta_1 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1)\}, \text{ and}$$

$$F_1 = \{q_0\};$$

$$\boxed{\mathcal{A}_2 = (\Sigma_2, S_2, S_2^0, \Delta_2, F_2):}$$

$$\Sigma_2 = \{a, b\},$$

$$S_2 = \{s_0, s_1, s_2\},$$

$$S_2^0 = \{s_0\},$$

$$\Delta_2 = \{(s_0, a, s_1), (s_0, b, s_2), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\}, \text{ and}$$

$$F_2 = \{s_2\}; \text{ and}$$

$$\boxed{\mathcal{A}_3 = (\Sigma_3, S_3, S_3^0, \Delta_3, F_3):}$$

$$\Sigma_3 = \{a, b\},$$

$$S_3 = \{s_0, s_1, s_2\},$$

$$S_3^0 = \{s_0\},$$

$$\Delta_3 = \{(s_0, a, s_1), (s_0, a, s_2), (s_1, b, s_0), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\}, \text{ and}$$

$$F_3 = \{s_2\}.$$

1. a) The union automaton  $\mathcal{A}_a$  built from  $\mathcal{A}_1$  and  $\mathcal{A}_2$  has the components

$$\boxed{\mathcal{A}_a = (\Sigma_a, S_a, S_a^0, \Delta_a, F_a):}$$

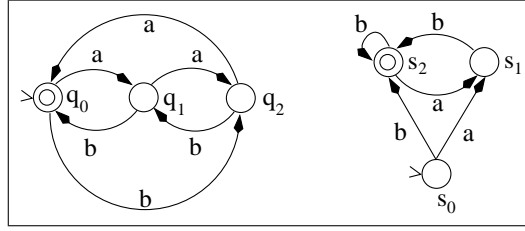
$$\Sigma_a = \{a, b\},$$

$$S_a = S_1 \cup S_2 = \{q_0, q_1, q_2, s_0, s_1, s_2\},$$

$$S_a^0 = S_1^0 \cup S_2^0 = \{q_0, s_0\},$$

$$\Delta_a = \Delta_1 \cup \Delta_2 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1), (s_0, a, s_1), (s_0, b, s_2), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\}, \text{ and}$$

$$F_a = F_1 \cup F_2 = \{q_0, s_2\}.$$



1. b) The product automaton  $\mathcal{A}_b$  built from  $\mathcal{A}_1$  and  $\mathcal{A}_2$  is

$$\mathcal{A}_b = (\Sigma_b, S_b, S_b^0, \Delta_b, F_b):$$

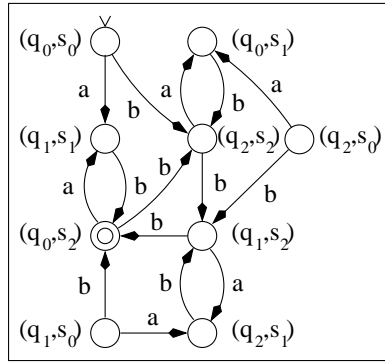
$$\Sigma_b = \{a, b\},$$

$$S_b = S_1 \times S_2 = \{(q_0, s_0), (q_0, s_1), (q_0, s_2), (q_1, s_0), (q_1, s_1), (q_1, s_2), (q_2, s_0), (q_2, s_1), (q_2, s_2)\},$$

$$S_b^0 = S_1^0 \times S_2^0 = \{(q_0, s_0)\},$$

$$\Delta_b = \left\{ \begin{aligned} &((q_0, s_0), a, (q_1, s_1)), ((q_0, s_0), b, (q_2, s_2)), ((q_0, s_1), b, (q_2, s_2)), \\ &((q_0, s_2), a, (q_1, s_1)), ((q_0, s_2), b, (q_2, s_2)), ((q_1, s_0), a, (q_2, s_1)), \\ &((q_1, s_0), b, (q_0, s_2)), ((q_1, s_1), b, (q_0, s_2)), ((q_1, s_2), a, (q_2, s_1)), \\ &((q_1, s_2), b, (q_0, s_2)), ((q_2, s_0), a, (q_0, s_1)), ((q_2, s_0), b, (q_1, s_2)), \\ &((q_2, s_1), b, (q_1, s_2)), ((q_2, s_2), a, (q_0, s_1)), ((q_2, s_2), b, (q_1, s_2)) \end{aligned} \right\},$$

$$F_b = F_1 \times F_2 = \{(q_0, s_2)\}.$$



1. c) Because  $((q_0, s_0), a, (q_1, s_1)) \in \Delta_b$ ,  $((q_1, s_1), b, (q_0, s_2)) \in \Delta_b$ ,  $(q_0, s_0) \in S_b^0$  and  $(q_0, s_2) \in F_b$  hold, the automaton  $\mathcal{A}_b$  has an accepting run  $(q_0, s_0), (q_1, s_1), (q_0, s_2)$  on the input  $ab \in \Sigma_b^*$ . Therefore  $ab \in L(\mathcal{A}_b) \neq \emptyset$  holds, and thus the language of  $\mathcal{A}_b$  is non-empty.

1. d) It is easy to see from the definition of  $\Delta_1$  that  $\{s' \in S_1 \mid (s, \sigma, s') \in \Delta_1\} \neq \emptyset$  holds for all  $s \in S_1$  and  $\sigma \in \Sigma_1$ , that is, the deterministic automaton  $\mathcal{A}_1$  has a completely specified transition relation. Therefore the automaton  $\mathcal{A}_d$  can be obtained from the automaton  $\mathcal{A}_1$  by taking

the complement of the set of  $\mathcal{A}_1$ 's accepting states with respect to  $S_1$ :  
formally,

$$\boxed{\mathcal{A}_d = (\Sigma_d, S_d, S_d^0, \Delta_d, F_d):}$$

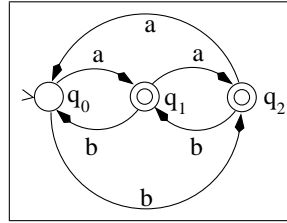
$$\Sigma_d = \Sigma_1 = \{a, b\},$$

$$S_d = S_1 = \{q_0, q_1, q_2\},$$

$$S_d^0 = S_1^0 = \{q_0\},$$

$$\Delta_d = \Delta_1 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1)\}, \text{ and}$$

$$F_d = S_1 \setminus F_1 = \{q_0, q_1, q_2\} \setminus \{q_0\} = \{q_1, q_2\}.$$



1. e) The deterministic automaton built from the automaton  $\mathcal{A}_3$  has the components

$$\boxed{\mathcal{A}_e = (\Sigma_e, S_e, S_e^0, \Delta_e, F_e):}$$

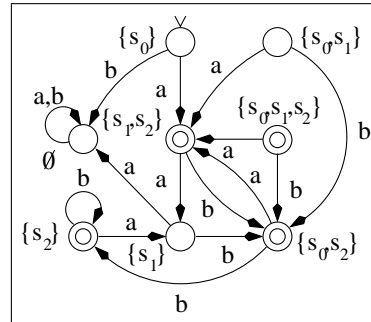
$$\Sigma_e = \Sigma_3 = \{a, b\},$$

$$S_e = 2^{S_3} = \{\emptyset, \{s_0\}, \{s_1\}, \{s_2\}, \{s_0, s_1\}, \{s_0, s_2\}, \{s_1, s_2\}, \{s_0, s_1, s_2\}\},$$

$$S_e^0 = \{S_3^0\} = \{\{s_0\}\},$$

$$\Delta_e = \{(\emptyset, a, \emptyset), (\emptyset, b, \emptyset), (\{s_0\}, a, \{s_1, s_2\}), (\{s_0\}, b, \emptyset), (\{s_1\}, b, \{s_0, s_2\}), (\{s_1\}, a, \emptyset), (\{s_2\}, a, \{s_1\}), (\{s_2\}, b, \{s_2\}), (\{s_0, s_1\}, a, \{s_1, s_2\}), (\{s_0, s_1\}, b, \{s_0, s_2\}), (\{s_0, s_2\}, a, \{s_1, s_2\}), (\{s_0, s_2\}, b, \{s_2\}), (\{s_1, s_2\}, a, \{s_1\}), (\{s_1, s_2\}, b, \{s_0, s_2\}), (\{s_0, s_1, s_2\}, a, \{s_1, s_2\}), (\{s_0, s_1, s_2\}, b, \{s_0, s_2\})\}, \text{ and}$$

$$F_e = \{s \in S_e \mid s \cap F_3 \neq \emptyset\} = \{\{s_2\}, \{s_0, s_2\}, \{s_1, s_2\}, \{s_0, s_1, s_2\}\}.$$



1. f) For all  $w \in \{a, b\}^*$ , let  $\#_a(w)$  and  $\#_b(w)$  denote the numbers of  $a$ 's and  $b$ 's in  $w$ , respectively. In this notation,

$$\begin{aligned} L(\mathcal{A}_1) &= \{w \in \{a, b\}^* \mid \#_a(w) \equiv \#_b(w) \pmod{3}\} \\ &= \{w \in \{a, b\}^* \mid \#_a(w) - \#_b(w) = 3k \text{ for some } k \in \mathbb{Z}\}. \end{aligned}$$

Formally, this result can be proved as follows. Let  $w = \sigma_1, \sigma_2, \dots, \sigma_n \in \{a, b\}^*$  be a word over the alphabet  $\{a, b\}$  for some  $n \geq 0$ ; because  $\mathcal{A}_1$  is a deterministic automaton with a completely specified transition relation, it is easy to see that  $\mathcal{A}_1$  has a unique run  $r = s_0, s_1, \dots, s_n$  on  $w$ .

We claim that for all  $0 \leq i \leq n$ ,  $s_i = q_j$  holds for some  $0 \leq j \leq 2$  such that  $\#_a(\sigma_1, \sigma_2, \dots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_i) = 3k + j$  for some  $k \in \mathbb{Z}$ . The result then follows from this claim because  $w \in L(\mathcal{A}_1)$  holds iff the run  $r$  is accepting iff  $s_n \in F_1 = \{q_0\}$  holds.

Because  $r$  is a run of  $\mathcal{A}_1$ ,  $s_0 \in S_1^0 = \{q_0\}$  holds, and because  $\#_a(\varepsilon) = \#_b(\varepsilon) = 0 = 3 \cdot 0$  holds<sup>1</sup>, the claim holds for  $i = 0$ .

Let  $0 \leq i < n$ , and let  $s_i = q_j$  for some  $0 \leq j \leq 2$ . Assume that  $\#_a(\sigma_1, \sigma_2, \dots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_i) = 3k + j$  holds for some  $k \in \mathbb{Z}$ . We show that the claim holds for  $i + 1$ .

If  $\sigma_{i+1} = a$  holds, then it is easy to see that  $\#_a(\sigma_1, \sigma_2, \dots, \sigma_{i+1}) = \#_a(\sigma_1, \sigma_2, \dots, \sigma_i) + 1$  and  $\#_b(\sigma_1, \sigma_2, \dots, \sigma_{i+1}) = \#_b(\sigma_1, \sigma_2, \dots, \sigma_i)$ . Therefore,

$$\begin{aligned} & \#_a(\sigma_1, \sigma_2, \dots, \sigma_{i+1}) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_{i+1}) \\ &= (\#_a(\sigma_1, \sigma_2, \dots, \sigma_i) + 1) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_i) \\ &= (\#_a(\sigma_1, \sigma_2, \dots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_i)) + 1 \\ &= \begin{cases} 3k + 1 & \text{for some } k \in \mathbb{Z} \text{ if } j = 0 \\ (3k + 1) + 1 = 3k + 2 & \text{for some } k \in \mathbb{Z} \text{ if } j = 1 \\ (3k + 2) + 1 = 3k + 3 = 3(k + 1) + 0 & \text{for some } k \in \mathbb{Z} \text{ if } j = 2 \end{cases} \\ &= 3k' + ((j + 1) \bmod 3) \text{ for some } k' \in \mathbb{Z}. \end{aligned}$$

On the other hand, it is easy to check from the transition relation of  $\mathcal{A}_1$  that  $s_{i+1} = q_{(j+1) \bmod 3}$  holds. Therefore, the claim holds for  $i + 1$  in this case.

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<sup>1</sup>Here,  $\varepsilon$  denotes the empty word over the alphabet  $\{a, b\}$ .

If  $\sigma_{i+1} = b$  holds, then

$$\begin{aligned}
& \#_a(\sigma_1, \sigma_2, \dots, \sigma_{i+1}) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_{i+1}) \\
&= \#_a(\sigma_1, \sigma_2, \dots, \sigma_i) - (\#_b(\sigma_1, \sigma_2, \dots, \sigma_i) + 1) \\
&= (\#_a(\sigma_1, \sigma_2, \dots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \dots, \sigma_i)) - 1 \\
&= \begin{cases} 3k - 1 = 3k - 3 + 2 = 3(k - 1) + 2 & \text{for some } k \in \mathbb{Z} \text{ if } j = 0 \\ (3k + 1) - 1 = 3k + 0 & \text{for some } k \in \mathbb{Z} \text{ if } j = 1 \\ (3k + 2) - 1 = 3k + 1 & \text{for some } k \in \mathbb{Z} \text{ if } j = 2 \end{cases} \\
&= 3k' + ((j + 2) \bmod 3) \text{ for some } k' \in \mathbb{Z},
\end{aligned}$$

and it is again easy to check from the transition relation that also  $s_{i+1} = q_{(j+2) \bmod 3}$  holds in this case.

The claim now follows by induction on  $i$  for all  $0 \leq i \leq n$ . □