Formally, the automata $A_1$, $A_2$, $A_3$ have the definitions

$$A_1 = (\Sigma_1, S_1, S_0^1, \Delta_1, F_1):$$
$$\Sigma_1 = \{a, b\},$$
$$S_1 = \{q_0, q_1, q_2\},$$
$$S_0^1 = \{q_0\},$$
$$\Delta_1 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1)\},$$
$$F_1 = \{q_0\};$$

$$A_2 = (\Sigma_2, S_2, S_0^2, \Delta_2, F_2):$$
$$\Sigma_2 = \{a, b\},$$
$$S_2 = \{s_0, s_1, s_2\},$$
$$S_0^2 = \{s_0\},$$
$$\Delta_2 = \{(s_0, a, s_1), (s_0, b, s_2), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\},$$
$$F_2 = \{s_2\};$$

$$A_3 = (\Sigma_3, S_3, S_0^3, \Delta_3, F_3):$$
$$\Sigma_3 = \{a, b\},$$
$$S_3 = \{s_0, s_1, s_2\},$$
$$S_0^3 = \{s_0\},$$
$$\Delta_3 = \{(s_0, a, s_1), (s_0, a, s_2), (s_1, b, s_0), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\},$$
$$F_3 = \{s_2\}. $$

1. a) The union automaton $A_a$ built from $A_1$ and $A_2$ has the components

$$A_a = (\Sigma_a, S_a, S_0^a, \Delta_a, F_a):$$
$$\Sigma_a = \{a, b\},$$
$$S_a = S_1 \cup S_2 = \{q_0, q_1, q_2, s_0, s_1, s_2\},$$
$$S_0^a = S_0^1 \cup S_0^2 = \{q_0, s_0\},$$
$$\Delta_a = \Delta_1 \cup \Delta_2 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1), (s_0, a, s_1), (s_0, b, s_2), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\},$$
$$F_a = F_1 \cup F_2 = \{q_0, s_2\}. $$
1. b) The product automaton $A_b$ built from $A_1$ and $A_2$ is
\[ A_b = (\Sigma_b, S_b, S_0^b, \Delta_b, F_b) : \]
\begin{align*}
\Sigma_b &= \{a, b\}, \\
S_b &= S_1 \times S_2 = \{(q_0, s_0), (q_0, s_1), (q_0, s_2), (q_1, s_0), (q_1, s_1), (q_1, s_2), (q_2, s_0), (q_2, s_1), (q_2, s_2)\}, \\
S_0^b &= S_1^0 \times S_2^0 = \{(q_0, s_0)\}, \\
\Delta_b &= \{(q_0, s_0, a, (q_1, s_1)), (q_0, s_0, b, (q_2, s_2)), (q_0, s_1, b, (q_2, s_2)), (q_0, s_1, a, (q_2, s_2)), (q_0, s_2, a, (q_1, s_1)), (q_0, s_2, b, (q_2, s_2)), (q_1, s_0, b, (q_2, s_2)), (q_1, s_0, a, (q_2, s_2)), (q_1, s_1, b, (q_2, s_2)), (q_1, s_1, a, (q_2, s_2)), (q_1, s_2, b, (q_0, s_2)), (q_1, s_2, b, (q_0, s_2)), (q_1, s_2, a, (q_0, s_1)), (q_2, s_0, b, (q_1, s_2)), (q_2, s_0, s, (q_1, s_2)), (q_2, s_1, b, (q_0, s_2)), (q_2, s_1, b, (q_0, s_2)), (q_2, s_1, s, (q_0, s_1)), (q_2, s_2, b, (q_1, s_2))\}, \\
F_b &= F_1 \times F_2 = \{(q_0, s_2)\}. 
\end{align*}

1. c) Because \((q_0, s_0, a, (q_1, s_1)) \in \Delta_b, ((q_1, s_1), b, (q_0, s_2)) \in \Delta_b, (q_0, s_0) \in S_b^0 \text{ and } (q_0, s_2) \in F_b\), the automaton $A_b$ has an accepting run \((q_0, s_0), (q_1, s_1), (q_0, s_2)\) on the input $ab \in \Sigma_b$. Therefore $ab \in L(A_b) \neq \emptyset$ holds, and thus the language of $A_b$ is non-empty.

1. d) It is easy to see from the definition of $\Delta_1$ that $\{s' \in S_1 \mid (s, \sigma, s') \in \Delta_1\} \neq \emptyset$ holds for all $s \in S_1$ and $\sigma \in \Sigma_1$, that is, the deterministic automaton $A_1$ has a completely specified transition relation. Therefore the automaton $A_d$ can be obtained from the automaton $A_1$ by taking
the complement of the set of $A_1$’s accepting states with respect to $S_1$; formally,

$$
\mathcal{A}_d = (\Sigma_d, S_d, S_d^0, \Delta_d, F_d): \\
\Sigma_d = \Sigma_1 = \{a, b\}, \\
S_d = S_1 = \{q_0, q_1, q_2\}, \\
S_d^0 = S_0^0 = \{q_0\}, \\
\Delta_d = \Delta_1 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), \\
(q_2, b, q_1)\}; \text{ and} \\
F_d = S_1 \setminus F_1 = \{q_0, q_1, q_2\} \setminus \{q_0\} = \{q_1, q_2\}.
$$

1. e) The deterministic automaton built from the automaton $A_3$ has the components

$$
\mathcal{A}_e = (\Sigma_e, S_e, S_e^0, \Delta_e, F_e): \\
\Sigma_e = \Sigma_3 = \{a, b\}, \\
S_e = 2^{S_3} = \emptyset, \{s_0\}, \{s_1\}, \{s_2\}, \{s_0, s_1\}, \{s_0, s_2\}, \{s_1, s_2\}, \{s_0, s_1, s_2\}, \\
S_e^0 = \{S_0^0\} = \{\{s_0\}\}, \\
\Delta_e = \{(\emptyset, a, \emptyset), (\emptyset, b, \emptyset), (\{s_0\}, a, \{s_1, s_2\}), (\{s_0\}, b, \emptyset), (\{s_1\}, b, \{s_1, s_2\}), \\
(\{s_1\}, a, \emptyset), (\{s_2\}, a, \{s_1\}), (\{s_2\}, b, \{s_2\}), (\{s_0, s_1\}, a, \{s_1, s_2\}), \\
(\{s_0, s_2\}, b, \{s_2\}), (\{s_1, s_2\}, a, \{s_1\}), (\{s_1, s_2\}, b, \{s_0, s_2\}), \\
(\{s_0, s_1, s_2\}, b, \{s_0, s_2\})\}; \text{ and} \\
F_e = \{s \in S_e \mid s \cap F_3 \neq \emptyset\} = \{\{s_2\}, \{s_0, s_2\}, \{s_1, s_2\}, \{s_0, s_1, s_2\}\}.
1. f) For all $w \in \{a, b\}^*$, let $\#_a(w)$ and $\#_b(w)$ denote the numbers of $a$’s and $b$’s in $w$, respectively. In this notation,

$$L(A_1) = \left\{ w \in \{a, b\}^* \mid \#_a(w) \equiv \#_b(w) \pmod{3} \right\}$$

$$= \left\{ w \in \{a, b\}^* \mid \#_a(w) - \#_b(w) = 3k \text{ for some } k \in \mathbb{Z} \right\}.$$  

Formally, this result can be proved as follows. Let $w = \sigma_1, \sigma_2, \ldots, \sigma_n \in \{a, b\}^*$ be a word over the alphabet $\{a, b\}$ for some $n \geq 0$; because $A_1$ is a deterministic automaton with a completely specified transition relation, it is easy to see that $A_1$ has a unique run $r = s_0, s_1, \ldots, s_n$ on $w$.

We claim that for all $0 \leq i \leq n$, $s_i = q_j$ holds for some $0 \leq j \leq 2$ such that $\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) = 3k + j$ for some $k \in \mathbb{Z}$. The result then follows from this claim because $w \in L(A_1)$ holds iff the run $r$ is accepting iff $s_n \in F_1 = \{q_0\}$ holds.

Because $r$ is a run of $A_1$, $s_0 \in S_1^0 = \{q_0\}$ holds, and because $\#_a(\varepsilon) = \#_b(\varepsilon) = 0 = 3 \cdot 0$ holds, the claim holds for $i = 0$.

Let $0 \leq i < n$, and let $s_i = q_j$ for some $0 \leq j \leq 2$. Assume that $\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) = 3k + j$ holds for some $k \in \mathbb{Z}$. We show that the claim holds for $i + 1$.

If $\sigma_{i+1} = a$ holds, then it is easy to see that $\#_a(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) = \#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) + 1$ and $\#_b(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) = \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i)$. Therefore,

$$\begin{align*}
\#_a(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) \\
= (\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) + 1) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) \\
= (\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i)) + 1 \\
3k + 1 & \quad \text{for some } k \in \mathbb{Z} \text{ if } j = 0 \\
= \begin{cases} 
(3k + 1) + 1 = 3k + 2 & \quad \text{for some } k \in \mathbb{Z} \text{ if } j = 1 \\
(3k + 2) + 1 = 3k + 3 = 3(k + 1) + 1 & \quad \text{for some } k \in \mathbb{Z} \text{ if } j = 2 \\
3k' + ((j + 1) \pmod{3}) & \quad \text{for some } k' \in \mathbb{Z}.
\end{cases}
\end{align*}$$

On the other hand, it is easy to check from the transition relation of $A_1$ that $s_{i+1} = q_{(j+1) \pmod{3}}$ holds. Therefore, the claim holds for $i + 1$ in this case.

$^1$Here, $\varepsilon$ denotes the empty word over the alphabet $\{a, b\}$.
If $\sigma_{i+1} = b$ holds, then

$$\#_a(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_{i+1})$$

$$= \#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - (\#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) + 1)$$

$$= (\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i)) - 1$$

$$= \begin{cases} 
3k - 1 = 3k - 3 + 2 = 3(k - 1) + 2 & \text{for some } k \in \mathbb{Z} \text{ if } j = 0 \\
(3k + 1) - 1 = 3k + 0 & \text{for some } k \in \mathbb{Z} \text{ if } j = 1 \\
(3k + 2) - 1 = 3k + 1 & \text{for some } k \in \mathbb{Z} \text{ if } j = 2 
\end{cases}$$

$$= 3k' + ((j + 2) \mod 3) \text{ for some } k' \in \mathbb{Z},$$

and it is again easy to check from the transition relation that also $s_{i+1} = q_{(j+2) \mod 3}$ holds in this case.

The claim now follows by induction on $i$ for all $0 \leq i \leq n$. \qed