T–79.4301 **Spring 2006**
Parallel and Distributed Systems
Tutorial 3 – Solutions

Formally, the automata $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ have the definitions

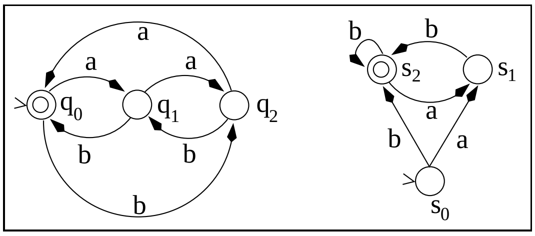
$\boxed{\mathcal{A}_1 = (\Sigma_1, S_1, S_1^0, \Delta_1, F_1):}$
$\Sigma_1 = \{a, b\}$,
$S_1 = \{q_0, q_1, q_2\}$,
$S_1^0 = \{q_0\}$,
$\Delta_1 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1)\}$, and
$F_1 = \{q_0\}$;

$\boxed{\mathcal{A}_2 = (\Sigma_2, S_2, S_2^0, \Delta_2, F_2):}$
$\Sigma_2 = \{a, b\}$,
$S_2 = \{s_0, s_1, s_2\}$,
$S_2^0 = \{s_0\}$,
$\Delta_2 = \{(s_0, a, s_1), (s_0, b, s_2), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\}$, and
$F_2 = \{s_2\}$; and

$\boxed{\mathcal{A}_3 = (\Sigma_3, S_3, S_3^0, \Delta_3, F_3):}$
$\Sigma_3 = \{a, b\}$,
$S_3 = \{s_0, s_1, s_2\}$,
$S_3^0 = \{s_0\}$,
$\Delta_3 = \{(s_0, a, s_1), (s_0, a, s_2), (s_1, b, s_0), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\}$, and
$F_3 = \{s_2\}$.

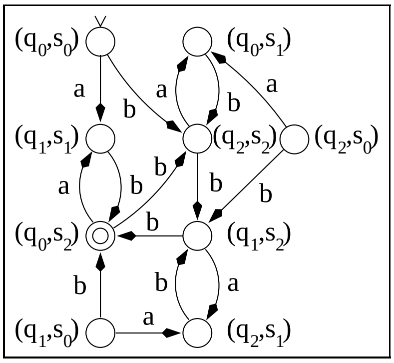
1. a) The union automaton $\mathcal{A}_a$ built from $\mathcal{A}_1$ and $\mathcal{A}_2$ has the components

$\boxed{\mathcal{A}_a = (\Sigma_a, S_a, S_a^0, \Delta_a, F_a):}$
$\Sigma_a = \{a, b\}$,
$S_a = S_1 \cup S_2 = \{q_0, q_1, q_2, s_0, s_1, s_2\}$,
$S_a^0 = S_1^0 \cup S_2^0 = \{q_0, s_0\}$,
$\Delta_a = \Delta_1 \cup \Delta_2 = \{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0), (q_2, b, q_1), (s_0, a, s_1), (s_0, b, s_2), (s_1, b, s_2), (s_2, a, s_1), (s_2, b, s_2)\}$, and
$F_a = F_1 \cup F_2 = \{q_0, s_2\}$.



1. b) The product automaton $\mathcal{A}_b$ built from $\mathcal{A}_1$ and $\mathcal{A}_2$ is

$$\boxed{\mathcal{A}_b = (\Sigma_b, S_b, S_b^0, \Delta_b, F_b):}$$
$$\Sigma_b = \{a, b\},$$
$$S_b = S_1 \times S_2 = \big\{(q_0,s_0),(q_0,s_1),(q_0,s_2),(q_1,s_0),(q_1,s_1),(q_1,s_2),(q_2,s_0),(q_2,s_1),(q_2,s_2)\big\},$$
$$S_b^0 = S_1^0 \times S_2^0 = \big\{(q_0,s_0)\big\},$$
$$\begin{aligned}\Delta_b = \Big\{ & \big((q_0,s_0),a,(q_1,s_1)\big), \big((q_0,s_0),b,(q_2,s_2)\big), \big((q_0,s_1),b,(q_2,s_2)\big),\\ & \big((q_0,s_2),a,(q_1,s_1)\big), \big((q_0,s_2),b,(q_2,s_2)\big), \big((q_1,s_0),a,(q_2,s_1)\big),\\ & \big((q_1,s_0),b,(q_0,s_2)\big), \big((q_1,s_1),b,(q_0,s_2)\big), \big((q_1,s_2),a,(q_2,s_1)\big),\\ & \big((q_1,s_2),b,(q_0,s_2)\big), \big((q_2,s_0),a,(q_0,s_1)\big), \big((q_2,s_0),b,(q_1,s_2)\big),\\ & \big((q_2,s_1),b,(q_1,s_2)\big), \big((q_2,s_2),a,(q_0,s_1)\big), \big((q_2,s_2),b,(q_1,s_2)\big)\Big\}, \text{ and}\end{aligned}$$
$$F_b = F_1 \times F_2 = \big\{(q_0,s_2)\big\}.$$

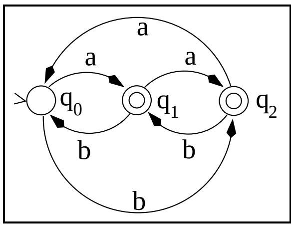


1. c) Because $\big((q_0,s_0),a,(q_1,s_1)\big) \in \Delta_b$, $\big((q_1,s_1),b,(q_0,s_2)\big) \in \Delta_b$, $(q_0,s_0) \in S_b^0$ and $(q_0,s_2) \in F_b$ hold, the automaton $\mathcal{A}_b$ has an accepting run $(q_0,s_0),(q_1,s_1),(q_0,s_2)$ on the input $ab \in \Sigma_b^*$. Therefore $ab \in L(\mathcal{A}_b) \neq \emptyset$ holds, and thus the language of $\mathcal{A}_b$ is non-empty.

1. d) It is easy to see from the definition of Δ_1 that $\{s' \in S_1 \mid (s,\sigma,s') \in \Delta_1\} \neq \emptyset$ holds for all $s \in S_1$ and $\sigma \in \Sigma_1$, that is, the deterministic automaton $\mathcal{A}_1$ has a completely specified transition relation. Therefore the automaton $\mathcal{A}_d$ can be obtained from the automaton $\mathcal{A}_1$ by taking

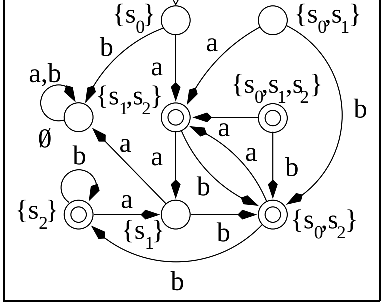
the complement of the set of $\mathcal{A}_1$'s accepting states with respect to S_1: formally,

$$\boxed{\mathcal{A}_d = (\Sigma_d, S_d, S_d^0, \Delta_d, F_d)\text{:}}$$
$$\begin{aligned}
\Sigma_d &= \Sigma_1 = \{a, b\},\\
S_d &= S_1 = \{q_0, q_1, q_2\},\\
S_d^0 &= S_1^0 = \{q_0\},\\
\Delta_d &= \Delta_1 = \big\{(q_0, a, q_1), (q_0, b, q_2), (q_1, a, q_2), (q_1, b, q_0), (q_2, a, q_0),\\
&\qquad (q_2, b, q_1)\big\}, \text{ and}\\
F_d &= S_1 \setminus F_1 = \{q_0, q_1, q_2\} \setminus \{q_0\} = \{q_1, q_2\}.
\end{aligned}$$



1. e) The deterministic automaton built from the automaton $\mathcal{A}_3$ has the components

$$\boxed{\mathcal{A}_e = (\Sigma_e, S_e, S_e^0, \Delta_e, F_e)\text{:}}$$
$$\begin{aligned}
\Sigma_e &= \Sigma_3 = \{a, b\},\\
S_e &= 2^{S_3} = \big\{\emptyset, \{s_0\}, \{s_1\}, \{s_2\}, \{s_0, s_1\}, \{s_0, s_2\}, \{s_1, s_2\}, \{s_0, s_1, s_2\}\big\},\\
S_e^0 &= \{S_3^0\} = \big\{\{s_0\}\big\},\\
\Delta_e &= \big\{(\emptyset, a, \emptyset), (\emptyset, b, \emptyset), \big(\{s_0\}, a, \{s_1, s_2\}\big), \big(\{s_0\}, b, \emptyset\big), \big(\{s_1\}, b, \{s_0, s_2\}\big),\\
&\qquad \big(\{s_1\}, a, \emptyset\big), \big(\{s_2\}, a, \{s_1\}\big), \big(\{s_2\}, b, \{s_2\}\big), \big(\{s_0, s_1\}, a, \{s_1, s_2\}\big),\\
&\qquad \big(\{s_0, s_1\}, b, \{s_0, s_2\}\big), \big(\{s_0, s_2\}, a, \{s_1, s_2\}\big), \big(\{s_0, s_2\}, b, \{s_2\}\big),\\
&\qquad \big(\{s_1, s_2\}, a, \{s_1\}\big), \big(\{s_1, s_2\}, b, \{s_0, s_2\}\big), \big(\{s_0, s_1, s_2\}, a, \{s_1, s_2\}\big),\\
&\qquad \big(\{s_0, s_1, s_2\}, b, \{s_0, s_2\}\big)\big\}, \text{ and}\\
F_e &= \{s \in S_e \mid s \cap F_3 \neq \emptyset\} = \big\{\{s_2\}, \{s_0, s_2\}, \{s_1, s_2\}, \{s_0, s_1, s_2\}\big\}.
\end{aligned}$$



1. f) For all $w \in \{a,b\}^*$, let $\#_a(w)$ and $\#_b(w)$ denote the numbers of a's and b's in w, respectively. In this notation,

$$\begin{aligned} L(\mathcal{A}_1) &= \big\{ w \in \{a,b\}^* \,\big|\, \#_a(w) \equiv \#_b(w) \pmod 3 \big\} \\ &= \big\{ w \in \{a,b\}^* \,\big|\, \#_a(w) - \#_b(w) = 3k \text{ for some } k \in \mathbb{Z} \big\}. \end{aligned}$$

Formally, this result can be proved as follows. Let $w = \sigma_1, \sigma_2, \ldots, \sigma_n \in \{a,b\}^*$ be a word over the alphabet $\{a,b\}$ for some $n \geq 0$; because $\mathcal{A}_1$ is a deterministic automaton with a completely specified transition relation, it is easy to see that $\mathcal{A}_1$ has a unique run $r = s_0, s_1, \ldots, s_n$ on w.

We claim that for all $0 \leq i \leq n$, $s_i = q_j$ holds for some $0 \leq j \leq 2$ such that $\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) = 3k + j$ for some $k \in \mathbb{Z}$. The result then follows from this claim because $w \in L(\mathcal{A}_1)$ holds iff the run r is accepting iff $s_n \in F_1 = \{q_0\}$ holds.

Because r is a run of $\mathcal{A}_1$, $s_0 \in S_1^0 = \{q_0\}$ holds, and because $\#_a(\varepsilon) = \#_b(\varepsilon) = 0 = 3 \cdot 0$ holds[1], the claim holds for $i = 0$.

Let $0 \leq i < n$, and let $s_i = q_j$ for some $0 \leq j \leq 2$. Assume that $\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) = 3k + j$ holds for some $k \in \mathbb{Z}$. We show that the claim holds for $i+1$.

If $\sigma_{i+1} = a$ holds, then it is easy to see that $\#_a(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) = \#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) + 1$ and $\#_b(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) = \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i)$. Therefore,

$$\begin{aligned} &\#_a(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) \\ &= \big(\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) + 1\big) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) \\ &= \big(\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i)\big) + 1 \\ &= \begin{cases} 3k+1 & \text{for some } k \in \mathbb{Z} \text{ if } j = 0 \\ (3k+1)+1 = 3k+2 & \text{for some } k \in \mathbb{Z} \text{ if } j = 1 \\ (3k+2)+1 = 3k+3 = 3(k+1)+0 & \text{for some } k \in \mathbb{Z} \text{ if } j = 2 \end{cases} \\ &= 3k' + \big((j+1) \bmod 3\big) \text{ for some } k' \in \mathbb{Z}. \end{aligned}$$

On the other hand, it is easy to check from the transition relation of $\mathcal{A}_1$ that $s_{i+1} = q_{(j+1) \bmod 3}$ holds. Therefore, the claim holds for $i+1$ in this case.

[1] Here, ε denotes the empty word over the alphabet $\{a,b\}$.

If $\sigma_{i+1} = b$ holds, then

$$
\begin{aligned}
&\#_a(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_{i+1}) \\
&= \#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \big(\#_b(\sigma_1, \sigma_2, \ldots, \sigma_i) + 1\big) \\
&= \big(\#_a(\sigma_1, \sigma_2, \ldots, \sigma_i) - \#_b(\sigma_1, \sigma_2, \ldots, \sigma_i)\big) - 1 \\
&= \begin{cases} 3k - 1 = 3k - 3 + 2 = 3(k-1) + 2 & \text{for some } k \in \mathbb{Z} \text{ if } j = 0 \\ (3k+1) - 1 = 3k + 0 & \text{for some } k \in \mathbb{Z} \text{ if } j = 1 \\ (3k+2) - 1 = 3k + 1 & \text{for some } k \in \mathbb{Z} \text{ if } j = 2 \end{cases} \\
&= 3k' + \big((j+2) \bmod 3\big) \text{ for some } k' \in \mathbb{Z},
\end{aligned}
$$

and it is again easy to check from the transition relation that also $s_{i+1} = q_{(j+2) \bmod 3}$ holds in this case.

The claim now follows by induction on i for all $0 \leq i \leq n$. $\square$