Lecture 7: Constraint satisfaction
Linear and integer programming

- Constraint satisfaction
  - Global constraints
  - Local search
  - Tools for SAT and CSP
- Linear and integer programming
  - Introduction

Global Constraints

- Constraint programming systems often offer constraints with special purpose constraint propagation (filtering) algorithms. Such a constraint can typically be seen as an encapsulation of a set of simpler constraints and is called a global constraint.
- A representative example is the alldiff constraint:

$$\mathrm{alldiff}(x_1,\ldots,x_n) = \{(d_1,\ldots,d_n) \mid d_i \neq d_j, \text{for } i \neq j\}$$

Example. A value assignment $\{x_1 \mapsto a, x_2 \mapsto b, x_3 \mapsto c\}$ satisfies $\mathrm{alldiff}(x_1,x_2,x_3)$ but $\{x_1 \mapsto a, x_2 \mapsto b, x_3 \mapsto a\}$ does not.

- $\mathrm{alldiff}(x_1,\ldots,x_n)$ can be seen as an encapsulation of a set of binary constraints $x_i \neq x_j$, $1 \leq i < j \leq n$.

Global Constraints: alldiff

- Global constraints enable compact encodings of problems.
- **Example.** N Queens
Problem: Place n queens on a $n \times n$ chess board so that they do not attack each other.
  - Variables: $x_1,\ldots,x_n$ (x_i gives the position of the queen on ith column)
  - Domains: [1..n]
  - Constraints: for $i \in [1..n-1]$ and $j \in [i+1..n]$:
(i) $\mathrm{alldiff}(x_1,\ldots,x_n)$ (rows)
(ii) $x_i - x_j \neq i - j$ (SW-NE diagonals)
(iii) $x_i - x_j \neq j - i$ (NW-SE diagonals)

Global Constraints: Propagation

- In addition to compactness global constraints often provide more powerful propagation than the same condition expressed as the set of corresponding simpler constraints.
- Consider the case of alldiff:
For $\mathrm{alldiff}(x_1,\ldots,x_n)$ there is an efficient hyper-arc consistency algorithm which allows more powerful propagation than hyper-arc consistency for the set of corresponding "$\neq$" constraints.
- **Example.**
  - Consider variables x_1,x_2,x_3 with domains $D_1 = \{a,b,c\}, D_2 = \{a,b\}, D_3 = \{a,b\}$.
  - Now $\mathrm{alldiff}(x_1,x_2,x_3)$ is not hyper-arc consistent and the projection rule removes values a,b from the domain of x_1.
  - However, the corresponding set of constraints $x_1 \neq x_2, x_1 \neq x_3, x_2 \neq x_3$ is hyper-arc consistent and the projection rule is not able to remove any values.

Global Constraints: Other Examples

- When solving a CSP problem often a special purpose (global) constraint and an efficient propagation algorithm for it needs to be developed to make the solution technique more efficient.
- There is a wide range of such global constraints (see for example Global Constraint Catalog `http://www.emn.fr/x-info/sdemasse/gccat/`):
  - cumulative
  - diff-n
  - cycle
  - sort
  - alldifferent and permutation
  - symmetric alldifferent
  - global cardinality (with cost)
  - sequence
  - minimum global distance
  - k-diff
  - number of distinct values

  ...

CSP: Local Search

- GSAT and WalkSAT type of local search algorithms (see Lecture 4) can be generalized to CSPs.
- As an example we consider Min Conflict Heuristic (MCH) algorithm (Minton et al, 1990):
  Given a CSP instance P
  - Initialize each variable by selecting a value uniformly at random from its domain.
  - In each local step select a variable x_i uniformly at random from the conflict set, which is the set of variables appearing in a constraint that is unsatisfied under the current assignment.
  - A new value v for x_i is selected from the domain of x_i such that by assigning v to x_i the number of conflicting constraints is minimized.
  - If there is more than one value with that property, one of the minimizing values is chosen uniformly at random.

Example.

Consider a run of MCH on a CSP
$\langle\{x_1 \leq x_2, x_2 \leq x_3, x_3 \leq x_1\}, x_1 \in \{1,2,3\}, x_2 \in \{1,2,3\}, x_3 \in \{1,2,3\}\rangle$

- First a value is selected for each variable uniformly at random from its domain, say $\{x_1 \mapsto 1, x_2 \mapsto 2, x_3 \mapsto 3\}$.
- For this assignment, the conflict set is $\{x_1, x_3\}$ from which, say, x_1 is randomly selected.
- Each possible assignment $x_1 \mapsto 1 / x_1 \mapsto 2 / x_1 \mapsto 3$ leaves one conflict and, hence, one of them is randomly selected, say $x_1 \mapsto 2$.
- For the resulting assignment $\{x_1 \mapsto 2, x_2 \mapsto 2, x_3 \mapsto 3\}$, the conflict set is $\{x_1, x_3\}$, from which x_3 is randomly selected.
- Now assignments $x_3 \mapsto 1 / x_3 \mapsto 3$ leave one conflict but $x_2 \mapsto 2$ leaves none.
- Hence, $x_2 \mapsto 2$ is selected leading to a solution $\{x_1 \mapsto 2, x_2 \mapsto 2, x_3 \mapsto 2\}$.

MCH—cont'd

- One can add to MCH a random walk step like in NoisyGSAT (WMCH algorithm; Wallace and Freuder, 1995).
- MCH can also be extended with a tabu search mechanism (Steinmann et al. 1997):
  - After each search step where the value of a variable x_i has changed from v to v', the assignment $x_i \mapsto v$ is declared tabu for the next tt steps.
  - While $x_i \mapsto v$ is tabu, value v is excluded from the selection of values for x_i except if assigning v to x_i leads to an improvement in the evaluation function over the current assignment.

CSP: Tabu Search

- A tabu search algorithm by Galiner and Hao is one of the best performing general local search algorithms for CSPs.
- TS-GH algorithm (Galiner and Hao, 1997):
  - Initialize each variable by selecting a value uniformly at random from its domain.
  - In each local step: among all variable-value assignments $x \mapsto v$ such that x appears in a constraint that is unsatisfied under the current assignment and v is in the domain of x, select an assignment $x \mapsto v$ that leads to the maximal decrease in the number of violated constraints.
  - If there are multiple such assignments, one of them is chosen uniformly at random.
  - After changing the assignment of x from v to v', the assignment $x \mapsto v$ is declared tabu for tt steps (except when leading to an improvement).
- For competitive performance, the evaluation function for variable-value assignments needs to be implemented using caching and incremental updating techniques.

Example.

Consider a local step of TS-GH on a CSP
$\langle\{x_1 \leq x_2, x_2 \leq x_3, x_3 \leq x_1\}, x_1 \in \{1,2,3\}, x_2 \in \{1,2,3\}, x_3 \in \{1,2,3\}\rangle$
where the current assignment is $\{x_1 \mapsto 2, x_2 \mapsto 2, x_3 \mapsto 3\}$

- Variables x_1, x_3 appear in an unsatisfiable constraint ($x_3 \leq x_1$).
- In MCH one of these would be randomly selected but in TS-GH we consider all assignments

$$x_1 \mapsto 1 / x_1 \mapsto 2 / x_1 \mapsto 3 / x_3 \mapsto 1 / x_3 \mapsto 2 / x_3 \mapsto 3$$

  and select an assignment leading to the maximal decrease in the number of violated constraints.
- Assignment $x_3 \mapsto 2$ leaves no violated constraints but other assignments leave a violated constraint.
- Hence, $x_3 \mapsto 2$ is selected leading to a solution $\{x_1 \mapsto 2, x_2 \mapsto 2, x_3 \mapsto 2\}$.

SAT: Local Search

- Local search methods have difficulties with structured problem instances.
- For good performance parameter tuning is essential. (For example in WalkSAT: the noise parameter p and the max_flips parameter.)
- Finding good parameter values is a non-trivial problem which typically requires substantial experimentation and experience.
- WalkSAT revised: adding greediness and adaptivity $\Longrightarrow$ Novelty+ and AdaptiveNovelty+ algorithms

function WalkSAT(F,p):
for max_tries times **do**
  $t \leftarrow$ initial truth assignment;
  while flips $<$ max_flips **do**
    if t satisfies F **then return** t **else**
      choose a random unsatisfied clause C in F;
      if some variables in C can be flipped without
        breaking any presently satisfied clauses,
        then pick one such variable x at random; else:
      with probability p, pick a variable x in C uniformly at random;
      with probability $(1-p)$, do basic GSAT move:
        find a variable x in C whose flipping causes
        largest decrease in the number of unsatisfied clauses ;
      $t \leftarrow (t$ with variable x flipped)
  end while;
end for
return "No satisfying truth assignment found"

Novelty+

- WalkSAT can be made greedier using a history-based variable selection mechanism.
- The age of a variable is the number of local search steps since the variable was last flipped.
- Novelty algorithm (McAllester et al., 1997): After choosing an unsatisfiable clause the variable to be flipped is selected as follows:
  - If the variable with the highest score does not have minimal age among the variables within the same clause, it is always selected.
  - Else it is only selected with probability $1-p$, where p is a parameter called noise setting.
  - Otherwise the variable with the next lower score is selected.
  - When sorting variables according to their scores, ties are broken according to decreasing age.
- In Novelty+ (Hoos 1998) a random walk step (with probability wp) is added: with probability $1-wp$ the variable to be flipped is selected according to the Novelty mechanism and in the other cases a random walk step is performed.

Adaptive WalkSat and Adaptive Novelty+

- A suitable value for the noise parameter p is crucial for competitive performance of WalkSAT and its variants.
- Too low noise settings lead to stagnation behaviour and too high settings to long running times.
- Instead of a static setting, a dynamically changing noise setting can be used in the following way:
- Two parameters θ and ϕ are given.
  - At the beginning the search is maximally greedy ($p = 0$).
  - There is a search stagnation if no improvement in the evaluation function value has been observed over the last $m\theta$ search steps where m is the number of clauses in the instance.
  - In this situation the noise value is increased by $p := p + (1-p)\phi$.
  - If there is an improvement in the evaluation function value, then the noise value is decreased by $p := p - p\phi/2$.

Adaptive WalkSat and Adaptive Novelty+

- Notice the asymmetry between increases and decreases in the noise setting.
- Between increases in noise level there is always a phase during which the search progress is monitored without further increasing the noise. No such delay is enforced between successive decreases in noise level.
- When this mechanism of adapting the noise level is applied to WalkSat and Novelty+, we obtain Adaptive WalkSat and Adaptive Novelty+ (Hoos, 2002).
- The performance of the adaptive versions is more robust w.r.t. the settings of θ and ϕ than the performance of the non-adaptive versions w.r.t. to the settings of p.
- For example, for Adaptive Novelty+ setting $\theta = 1/6$ and $\phi = 0.2$ seem to lead to robust overall performance (while there appears to be no such setting for p in the non-adaptive case).

Tools for SAT

- The development of SAT solvers is strongly driven by SAT competitions (`http://www.satcompetition.org/`)
- There is a wide range of efficient solvers also available in public domain.
- See for example `http://www.satcompetition.org/` for solvers that ranked well in previous SAT competitions.

  SAT 2005:

  `SatELiteGTI, MiniSAT 1.13, zChaff_rand, HaifaSAT, Vallst, March_dl, kcnf-2004, Dew_Satz1a, Jerusat 1.31 B,`

  SAT-Race 2006:

  `minisat 2.0, Eureka 2006, Rsat, Cadence MiniSat v1.14, ...`

  SAT 2007:

  `minisat, SATzilla, MiraXT, Rsat, picosat, March KS, adaptg2wsat+, adaptg2wsat0, MXT, KCNFS 2004, ...`

Tools for CSP

- Constraint programming systems offer a rich set of supported constraint types with efficient propagation algorithms and primitives for implementing search.
- Typically the user needs to program, for example, the search algorithm, splitting technique, and heuristic.
- See, for example, `http://4c.ucc.ie/web/archive/solver.jsp` for available constraint solvers:

  `CLAIRE, ECLiPse, GNU Prolog, Oz,`
  `Sicstus Prolog, ILOG Solver, ...`

Linear and Integer Programming

- Linear and Integer Programming can be thought to be a subclass of constraint programming where there are
  - two types of variables: real valued and integer valued
  - only one type of constraint: linear (in)equalities.
- Linear Programming (LP): only real valued variables.
- Integer Programming (IP): only integer variables.
- Mixed Integer Programming (MIP): both integer and real valued variables.

Linear and Integer Programming

- Computationally there is a fundamental difference between LP and IP:
  LP problems can be solved efficiently (in polynomial time) but IP problems are NP-complete (and all known algorithms have an exponential worst-case running time).
- MIP offers an attractive framework for solving (search and) optimization problems:
  - Continuous variables can be handled efficiently along with discrete variables.
  - Powerful LP solution techniques can be exploited in the IP case through linear relaxation.
  - Bounds on deviation from optimality can be generated even when optimal solutions are not proven.

MIP: Basic Concepts

- In a mixed integer program (MIP) variables are partitioned in two sets such that in the other set (call this **I**) each variable is required to take an integer value while the remaining variables can take any real value.
- Each variable x_i can have a range $l_i \leq x_i \leq u_i$.
- A linear constraint is an expression of the form

  $$a_1x_1 + \cdots + a_nx_n = b$$

  where the relation symbol '$=$' can also be '$\leq$' or '$\geq$' and a_i and b are given constants.
- A linear function is an expression of the form $c_1x_1 + \cdots + c_nx_n$
- A MIP consists of (i) the objective of minimizing (or maximizing) a linear function, (ii) a set of linear constraints, (iii) ranges for variables and (iv) a set of integer valued variables.

An Example MIP

$\min x_2 - x_1$ s.t.

$$\begin{aligned} 3x_1 &- x_2 \geq 0 \\ x_1 &+ x_2 \geq 6 \\ -x_1 &+ 2x_2 \geq 0 \\ 2 \leq\ & x_1 \leq 10 \end{aligned}$$

x_2 is integer

MIP: Basic Concepts

- We can write a MIP in the matrix form as follows.
- Let x be a vector of variables $x = (x_1, \ldots, x_n)$.
- Variable ranges can be represented by vectors $l = (l_1, \ldots, l_n)$ and $u = (u_1, \ldots, u_n)$ such that for all i, $l_i \leq x_i \leq u_i$, that is, $l \leq x \leq u$.
- A set of linear constraints $\Sigma_j a_j x_j = b_j$ can be written in matrix form as $Ax = b$ such that $A = (a_{ij})$ is a matrix where a_{ij} is the coefficient for variable j in the ith constraint and $b = (b_1, \ldots, b_n)$.
- A linear objective function $\Sigma_j c_j x_j$ is written as cx where $c = (c_1, \ldots, c_n)$ is a vector of coefficients.
- Then a MIP can be written as:

$$\begin{aligned} & \min cx \\ s.t.\quad & Ax = b \\ & l \leq x \leq u \\ & x_j \text{ is integer for all } j \in \mathbf{I} \end{aligned}$$

MIP: Basic Concepts

- A feasible solution to a MIP is an assignment of values to the variables in the problem such that the assignment satisfies all the linear constraints and range constraints and for each variable in I it assigns an integer value.
- A program is feasible if it has a feasible solution otherwise it is said to be infeasible.
- An optimal solution is a feasible solution that gives the minimal (maximal) value of the objective function among all feasible solutions.
- A program is unbounded (from below) if for all $\lambda \in R$ there is a feasible solution for which the value of the objective function is at most λ.

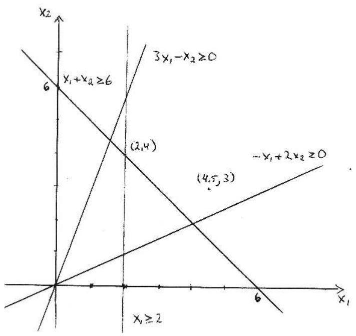
An Example

Consider the MIP

$\min 2x_1 + x_2$ s.t.

$$\begin{aligned} 3x_1 &- x_2 \geq 0 \\ x_1 &+ x_2 \geq 6 \\ -x_1 &+ 2x_2 \geq 0 \\ 2 \leq\ & x_1 \end{aligned}$$

x_2 is integer



- $x_1 = 4.5$, $x_2 = 3$ is a feasible solution
- $x_1 = 2$, $x_2 = 4$ is an optimal solution which gives the minimal value (8) for the objective function.
- If the objective is $\min x_1 - x_2$, then the problem is unbounded (from below).
- If we change the range for x_1 to be $x_1 \leq 1$, the problem becomes infeasible.

Modelling: SET COVER

INSTANCE: A family of sets $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set U.
QUESTION: Find an l-cover of U (a set of l sets from F whose union is U) with the smallest number l of sets.

- For each set S_i an integer variable x_i such that $0 \leq x_i \leq 1$
- For each element u of U a constraint

$$a_1x_1 + \cdots + a_nx_n \geq 1$$

  where the coefficient $a_i = 1$ if $u \in S_i$ and otherwise $a_i = 0$.
- Objective: $\min x_1 + \cdots + x_n$

Modelling: Logical Constraints

- Consider binary integer variables ($0 \leq x_i \leq 1$).
- Disjunction: x_3 has the value of the boolean expression $x_1 \vee x_2$:

$$x_3 \geq x_1$$
$$x_3 \geq x_2$$
$$x_3 \leq x_1 + x_2$$

- Conjunction: x_3 has the value of the boolean expression $x_1 \wedge x_2$:

$$x_3 \leq x_1$$
$$x_3 \leq x_2$$
$$x_3 \geq x_1 + x_2 - 1$$

Modelling SAT

Given a SAT instance F in CNF, introduce

- for each Boolean variable x in F, a binary integer variable x ($0 \leq x \leq 1$).
- for each clause $l_i \vee \cdots \vee l_n$ in F, a constraint

$$a_1x_1 + \cdots + a_nx_n \geq 1 - m$$

  where the coefficient $a_i = 1$ if the literal l_i is positive and otherwise $a_i = -1$ and m is the number of negative literals in the clause.
- Then F is satisfiable iff the corresponding set of constraints has a feasible solution.