# Lecture 2: Combinatorial search and optimisation problems

- Different types of computational problems
- Examples of computational problems
- Relationships between problems
- Computational properties of different problems.





## Computational problems

- A (computational) problem: an infinite set of possible instances with a question.
- A decision problem: a question with a yes/no answer

### Example

REACHABILITY
INSTANCE: A graph $(V, E)$ and nodes $v, u \in V$.
QUESTION: Is there a path in the graph from $v$ to $u$?





## Computational problems

Often more complicated questions are of interest:

- Search (function) problem:
  given an instance find a solution (object satisfying certain properties).
- Optimization problem:
  given an instance find a best solution according to some cost criterion.
  Typically this is formalized by specifying
  - what are feasible solutions for an instance and
  - a cost function which assigns a cost (typically a integer/real number) to each feasible solution.

  Now a solution to an optimization problem instance is a feasible solution that has the minimal (or maximal) cost.
- Counting problem:
  given an instance count the number of solutions.





## Examples

- PATH
  INSTANCE: A graph $(V, E)$ and nodes $v, u \in V$.
  QUESTION: Find a path from $v$ to $u$.
- SHORTEST PATH
  INSTANCE: A graph $(V, E)$ and nodes $v, u \in V$.
  QUESTION: Find a shortest path from $v$ to $u$.
- #PATH
  INSTANCE: A graph $(V, E)$ and nodes $v, u \in V$.
  QUESTION: Count the number of simple paths from $v$ to $u$.

## Easy and hard problems

- Many problems are computationally easy: there is a polynomial time algorithm for the problem, i.e. there is an algorithm solving the problem whose run time increases polynomially w.r.t. the size of the input instance. Consider, e.g., REACHABILITY.
- Some problems are not computationally easy: there is no known guaranteed polynomial time algorithm for the problem, i.e. for any known algorithm there is an infinite collection of instances for which the run time increases super-polynomially w.r.t. the size of the instance.
- This course focuses on methods for solving such problems in practice.





## Examples of hard problems

- SAT
  INSTANCE: a propositional formula in conjunctive normal form
  QUESTION:
  (D) Is the formula satisfiable?
  (S) Find a satisfiable truth assignment for the formula.
  (O) Find a truth assignment that satisfies the most clauses in the formula.
- GRAPH COLORING
  INSTANCE: A graph $(V, E)$ and a positive integer $k$
  QUESTION:
  (D) Is there a $k$-coloring of the graph, i.e. an assignment of one of the $k$ colors to each vertex such that vertices connected with an edge do not have the same color?
  (S) Find a $k$-coloring.
  (O) Find an $l$-coloring with the smallest number $l$ of colors.





## Examples of hard problems (II)

- CLIQUE
  INSTANCE: A graph $(V, E)$ and a positive integer $k$
  QUESTION:
  (D) Is there a $k$-clique in the graph, i.e. a set of $k$ nodes such that there is an edge between every pair of vertices from the set.
  (S) Find a $k$-clique.
  (O) Find an $l$-clique with the largest number $l$ of vertices.
- SET COVER
  INSTANCE: A family of sets $F = \{S_1, \ldots, S_n\}$ of subsets of a finite set $U$ and a positive integer $k$.
  QUESTION:
  (D) Is there $k$-cover of $U$, i.e., a set of $k$ sets from $F$ whose union is $U$.
  (S) Find a $k$-cover of $U$.
  (O) Find a set $l$-cover of $U$ with the smallest number $l$ of sets.





## Examples of hard problems (III)

TSP (TRAVELING SALESPERSON)
INSTANCE: $n$ cities $1, \ldots, n$ and a nonnegative integer distance $d_{ij}$ between any two cities $i$ and $j$ (such that $d_{ij} = d_{ji}$) and a positive integer $B$.
QUESTION:
(D) Is there a tour of length at most $B$, i.e. a permutation $\pi$ of the cities such that the length

$$\sum_{i=1}^{n} d_{\pi(i)\pi(i+1)}$$

is at most $B$ (where $\pi(n+1) = \pi(1)$)?
(S) Find a tour of length at most $B$.
(O) Find the shortest tour of the cities.

## Relationship between problems

- An interesting relationship between two computational problems $A$ and $B$ is that of a reduction.
- $B$ reduces to $A$ ($B \sqsubseteq A$) if there is a transformation $R$ which for every input instance $x$ of $B$ produces an equivalent input instance $R(x)$ of $A$ (where equivalent means that the answer (yes/no) for $R(x)$ considered as the input of $A$ is the correct answer to $x$ as an input of $B$).
- For a reduction to be useful in needs to be relatively easy to compute (compared to the problems $A$ and $B$).
- Typically it is assumed that the reduction can be computed in polynomial time.

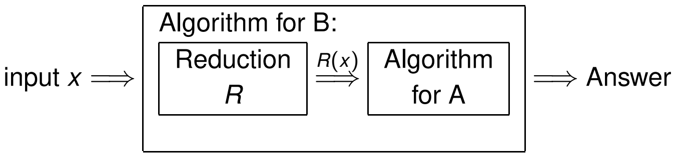




## Reduction

Reduction from $B$ to $A$ ($B \sqsubseteq A$) can be exploited in two interesting ways:

- an algorithm for $B$ can be built on top of an algorithm for $A$.
- reduction implies that $A$ is computationally at least as hard as $B$.




- The former is used extensively in the course.
- The latter is used in computational complexity theory (T-79.5104) to classify computational problems; $B \sqsubseteq A$ orders problems by difficulty.





## Example: 3-COL $\sqsubseteq$ SAT

- 3-COL
  INSTANCE: a graph $(V, E)$.
  QUESTION: is there a 3-coloring of the graph.
- Reduction from 3-COL to SAT

  | For each vertex $v \in V$: | For each edge $(v, u) \in E$: |
  | --- | --- |
  | $v(1) \vee v(2) \vee v(3)$ | $\neg v(1) \vee \neg u(1)$ |
  | $\neg v(1) \vee \neg v(2)$ | $\neg v(2) \vee \neg u(2)$ |
  | $\neg v(1) \vee \neg v(3)$ | $\neg v(3) \vee \neg u(3)$ |
  | $\neg v(2) \vee \neg v(3)$ | |

- This is a reduction because
  (i) it can be computed efficiently and
  (ii) it produces from an instance of 3-COL an equivalent instance of SAT: the graph has a 3-coloring iff the set of clauses is satisfiable.





## Example: 3-SAT $\sqsubseteq$ INDEPENDENT SET

- INDEPENDENT SET
  INSTANCE: A graph $G = (V, E)$ and an integer $K$.
  QUESTION: Is there an independent set $I \subseteq V$ with $|I| = K$.
  (A set $I \subseteq V$ is independent if $i, j \in I$ implies that there is no edge between $i$ and $j$).
- Reduction from 3-SAT to INDEPENDENT SET
  Given a set $\phi$ of $m$ clauses each with three literals, construct a graph whose vertices are the occurrences of the literals in $\phi$ and add edges so that for each clause there is a separate triangle and then add an edge between two vertices in different triangles if they correspond to complementary literals.
  Finally, set $K = m$.

## Example: 3-SAT $\sqsubseteq$ INDEPENDENT SET—cont'd

- This is a reduction because $\phi$ is satisfiable iff there is an independent set of size $m$ for the graph.
  ($\Rightarrow$) If $\phi$ has a satisfying truth assignment, then take one vertex from each triangle for which the corresponding literal is true in the assignment and this gives an independent set of size $m$.
  ($\Leftarrow$) If there is an independent set of size $m$, then it contains exactly one vertex from each triangle and no two vertices corresponding to complementary literals. Hence, the set induces a truth assignment for which each clause has a true literal implying that $\phi$ is satisfiable.





## Example: INDEPENDENT SET $\sqsubseteq$ CLIQUE

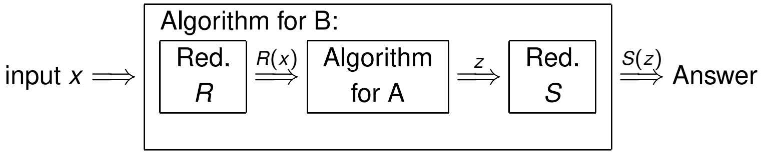
- Reduction from INDEPENDENT SET to CLIQUE
  Given a $G = (V, E)$ and an integer $K$, take the complement graph $G' = (V, \{(v, u) \mid v, u \in V, (v, u) \notin E\}$.
- This is a reduction because an independent set of a graph is a clique of the complement graph.
- Reductions compose (are transitive):
  3-SAT $\sqsubseteq$ INDEPENDENT SET and
  INDEPENDENT SET $\sqsubseteq$ CLIQUE imply
  3-SAT $\sqsubseteq$ CLIQUE
- Hence, using an algorithm for CLIQUE, we can solve INDEPENDENT SET, 3-SAT, 3-COL using reductions.





## Reductions—cont'd

- Reductions for search problems need a translation of the result back to the original problem:
  A reduction from a search problem $B$ to $A$ is a pair of mappings $(R, S)$ (both computable in polynomial time) such that for all $x, z$: if $x$ is an instance of $B$, then $R(x)$ is an instance of $A$ and if $z$ is a correct output of $R(x)$, then $S(z)$ is a correct output of $x$.
- For optimization problems optimality needs to be preserved, too.








## Size of the reductions

In practice not all polynomial time reductions are useful in building algorithms on top of others but the size of the translation matters.

### Example

- Consider a problem $A$ for which we have a $2^{n/1000}$ algorithm. Hence, an input of length n=20000 needs $2^{20000/1000} \approx 10^6$ steps.
- We want to use this algorithm to solve a difficult problem $B$ for which we have a quadratic translation to $A$.
- Now the run time of the combined algorithm for $B$ is $p(n) + 2^{n^2/1000}$ where $p(n)$ is a polynomial giving the run time of the translation from $B$ to $A$.
- For an input of length n=20000 the run time is $p(20000) + 2^{20000^2/1000} \geq 2^{400000} \geq 10^{10000}$ steps!

## Relationship between different kinds of problems

Decision problems vs search problems

- A decision problem reduces to the corresponding search problem trivially, i.e., if a search problem can be solved efficient so can the corresponding decision problem.
- But also a search problem reduces to the corresponding decision problem.





## SET COVER(D) vs SET COVER(S)

- If SET COVER(S) can solved in polynomial time, then so can SET COVER(D).
- If SET COVER(D) can solved in polynomial time, then so can SET COVER(S) using the following algorithm given a family $F = \{S_1, \ldots, S_n\}$ of subsets of $U$ and a positive integer $k$.

  if $F$ does not have a $k$-cover then return "no";
  l := k-1;
  for all $S \in \{S_1, \ldots, S_n\}$ do
  if $F[S := \textbf{true}]$ has an $l$-cover then
  $T(S) := \textbf{true}$; $F := F[S := \textbf{true}]$; l := l - 1
  else $T(S) := \textbf{false}$ ; $F := F[S := \textbf{false}]$;
  return $T$;

  where $\{S \in F \mid T(S) = \textbf{true}\}$ is the computed k-cover;
  $F[S := \textbf{true}]$ denotes the family $F$ with the set $S$ and its elements removed from $F$ and $U$;
  $F[S := \textbf{false}]$ is just the set $S$ removed from $F$.





## Decision vs optimization problems

Consider TSP(D) vs TSP(O)

- If TSP(O) can solved in polynomial time, then so can TSP(D).
- If TSP(D) can solved in polynomial time, then so can TSP(O).
- An optimal tour can be found using an algorithm which
  1. finds the cost $C$ of an optimal tour by binary search (with a polynomial number of calls to the polynomial time algorithm for TSP(D));
  2. finds an optimal tour using $C$ (with a polynomial number of calls to the polynomial time algorithm for TSP(D)).





## TSP(D) vs TSP(O)

A TSP(O) algorithm using a TSP(D) algorithm as a subroutine:

/*Find the cost $C$ of an optimal tour by binary search*/
$C := 0$; $C_u := 2^n$;
while $(C_u > C)$ do
if there is a tour of cost $\lfloor (C_u + C)/2 \rfloor$ or less then
$C_u := \lfloor (C_u + C)/2 \rfloor$
else $C := \lfloor (C_u + C)/2 \rfloor + 1$;
/* Find an optimal tour */
For all intercity distances do
set the distance to $C + 1$;
if there is a tour of cost $C$ or less, freeze the distance to $C + 1$
else restore the original distance and add it to the tour;
endfor

## Different kinds of optimization problems

- Consider the traveling salesperson problem and two new variants:
  EXACT TSP: Given a distance matrix and an integer $B$, is the length of the shortest tour equal to $B$?
  TSP COST: Given a distance matrix, compute the length of the shortest tour.
- It can be shown that the four variants can be ordered in "increasing complexity" by reductions:
  TSP(D) ; EXACT TSP; TSP COST; TSP(O)
- All the four variants of TSP are polynomially equivalent: there is a polynomial-time algorithm for one iff there is one for all four (because TSP(D) and TSP(O) are).





## Computational properties of problems

- The previous arguments indicate that for a problem the decision, search and optimization variants are polynomially equivalent.
- However, this does not imply that they are equally easy to solve in practice.
- There are differences if no polynomial algorithm is known.
- For a decision problem the "yes" answer is often easy to verify.
  - Typically, the question is about existence of a certain objects (witness/certificate) such as a satisfying truth assignment, a coloring, . . .
  - If the witness is given, then the correctness of the "yes" answer can be checked in polynomial time.
  - However, the "no" answer is more challenging to verify because there is no obvious witness/certificate for the answer, e.g., for the lack of coloring.





## Computational properties of problems (II)

- The same holds for search problems where the correctness of the found object can typically be checked in polynomial time but where the "no" answer is more challenging to verify.
- Notice that even if the verification of a solution is easy, this does not imply that finding a solution is easy.
- Many engineering problems fall into this class of problems
  - A typical problem is to construct a mathematical object satisfying certain specifications (path, solution of equations, routing, VLSI layout,. . . ). This is the certificate.
  - The decision version of the problem is determine whether at least one such an object exists for the input.
  - The object is usually not very large compared to the input.
  - The specifications of the object are usually simple enough to be checkable in polynomial time.





## Computational properties of problems (III)

- The decision versions of this class of problems form the problem class NP, i.e., decision problems with polynomial size certificates that are checkable in polynomial time.
- The hardest problems in this class (w.r.t. $\sqsubseteq$) are called NP-complete problems and they include, for example, SAT, GRAPH COLORING, CLIQUE, SET COVER, TSP, . . .
- To learn more, see computational complexity theory, for example, course T-79.5104 in the autumn term.
- For optimization problems it is hard even to verify a solution.
  - Consider an instance of the traveling salesperson problem and its potential solution $\pi$.
  - There seems to be no obvious polynomial time test that could establish that $\pi$ is actually a tour of the cities that has the shortest possible length.
- Counting problems are often even harder.

## Computational properties of optimization problems

- The computational hardness of verifying a solution depends on the type of an optimization problem.
- EXACT TSP: checking whether the length of the shortest tour equals to $B$ requires two calls to the decision problem:
  - check whether there is a tour of length at most $B$?
  - check whether there is not a tour of length at most $B-1$?
- However, checking the length of the shortest tour seems to require polynomial number of adaptive calls to the decision procedure (see binary search above).
- The same holds for checking the shortest tour.



T–79.4201 Search Problems and Algorithms

## Algorithm design techniques for hard problems

- There are several approaches to developing efficient algorithms for computationally challenging problems such as:
  - identify special cases (using tools from complexity theory) and develop special algorithms for these
  - approximation algorithms
  - randomized algorithms
- However, it typically requires a substantial amount of expertise and resources to develop an efficient algorithm for a problem.
- For example, in practical applications it often happens that the problem specification is not “mathematically clean” but includes a number of “side conditions” and criteria which are fairly complicated to integrate into an algorithm. Moreover, these “side conditions” tend to change quite frequently.
- In this course we study search algorithms as a practical set of tools to solve such problems.

I.N. & P.O. Spring 2006