Photon Polarization
T-79.4001 Seminar on Theoretical Computer Science

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Outline

Introduction
  Quantum Mechanics
  Two-dimensional Real Linear Algebra

Photon Polarization
  Linear Polarization
  Review of Complex Numbers
  Circular and Elliptical Polarization
Quantum Mechanics

- Developed between 1900 and 1926
- Radically changed our understanding of the physical world
  - Introduced probabilistic (undeterministic) behavior
  - It allows mutually exclusive situations to exist simultaneously in quantum superposition.
- We need to mathematically represent quantum
  - states
  - measurements
  - reversible transformations
  - (composite systems)
- My focus: photons and their polarization
Real Vectors 1/2

- **Definition**
  A column vector: \( |s\rangle = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \)

- **Definition**
  A row vector: \( \langle s| = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \)

- **Definition**
  The dot/inner product between two real vectors \( |s\rangle \) and \( |m\rangle \):

\[
\langle s|m\rangle = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = s_1 m_1 + s_2 m_2
\]
Real Vectors 2/2

Definition
A vector is **normalized** if its length is 1.

Example
The following vectors are normalized: \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \)

Definition
Two vectors \(|s\rangle\) and \(|m\rangle\) are **orthogonal** (i.e. perpendicular) iff their inner product \(\langle s | m \rangle\) is 0.

Example
The following vectors are mutually orthogonal:

\[
\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \cdot 0 + 0 \cdot 1 = 0
\]
Representing Linear Polarization States

- A photon’s linearization state is described by an axis on a plane.
- Such an axis is mathematically described by a normalized vector. The general form of a normalized vector is
  \[
  \begin{pmatrix}
  \cos \theta \\
  \sin \theta 
  \end{pmatrix}
  \]
- Vector \(|s\rangle\) and its inverse \(-|s\rangle\) represent the same linear polarization state.
Measurements

- There can be no device that directly ‘reads’ the polarization state of a photon.
- Instead, there are only devices that ‘ask’ photons binary questions. More accurately, such a device can have a photon choose between two mutually orthogonal states.
  - e.g. “Which are you: vertically or horizontally polarized?”
- Even if the photon is in neither state, it still has to make a choice between the two states. The choice is done in a probabilistic fashion.
- The photon’s state becomes the state that it chooses.
A polarization measurement device can be mathematically represented by an ordered pair of mutually orthogonal normalized state vectors (i.e. an orthonormal basis):

\[ M = (|m^{(1)}\rangle, |m^{(2)}\rangle) \]

When a photon with polarization state \(|s\rangle\) is subjected to measurement \(M\), it probabilistically chooses between \(|m^{(1)}\rangle\) and \(|m^{(2)}\rangle\). The probability of the photon choosing the state \(|m^{(i)}\rangle\) is \(|\langle s|m^{(i)}\rangle|^2\).

So, the photon is much more likely to choose the state \(|m^{(i)}\rangle\) that is closer to its original state \(|s\rangle\).
The Math of Measurements 2/2

Example (A measurement)

Let
\[ |m^{(1)}\rangle = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), \]
\[ |m^{(2)}\rangle = \left( \frac{\sqrt{3}}{2}, -\frac{1}{2} \right), \] and
\[ |s\rangle = \left( \begin{array}{c} 1 \\ 0 \end{array} \right). \]

What are the probabilities of the two outcomes?

\[ p_1 = |\langle s | m^{(1)} \rangle|^2 = \left( \frac{1}{2} \right)^2 = 1/4 \]
\[ p_2 = |\langle s | m^{(2)} \rangle|^2 = \left( \frac{\sqrt{3}}{2} \right)^2 = 3/4 \]

As expected \( p_1 + p_2 = 1. \)
Polarization Measurement Devices in Practise

- **Wollaston prism**
  - Takes a beam of unpolarized light and divides it into separate horizontally and vertically polarized beams.

- **Polarizing filter**
  - Like a wollaston prism, but polarizes part of the beam and simply absorbs the rest.

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*Figure: A Wollaston prism*
Reversible Transformations

- A photon’s polarization state can be reversibly transformed.
- Two types: (i) rotations and (ii) reflections around an axis.
- The allowed transformations are mathematically represented by $2 \times 2$ real orthogonal matrices.
  - A matrix $R$ is orthogonal iff $RR^T = R^T R = I$, where $I$ is the identity matrix.
  - Let $|s\rangle$ be a polarization state and $R$ be a transformation matrix. Then the outcome of transformation $R$ on the state $|s\rangle$ is $R|s\rangle$.

  - General rotation matrix: $$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$
  - Reflection across a line of given angle: $$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$
Transformation examples

Example (A rotation)

Let’s rotate a horizontal state vector \( |s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) by 45° CCW:

\[
\begin{pmatrix}
\cos 45° & -\sin 45° \\
\sin 45° & \cos 45° \\
\end{pmatrix}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}
\]

Example (A reflection)

Let’s reflect the same vector \( |s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) around an axis with the angle 45°:

\[
\begin{pmatrix}
\cos(2 \cdot 45°) & \sin(2 \cdot 45°) \\
\sin(2 \cdot 45°) & -\cos(2 \cdot 45°) \\
\end{pmatrix}
\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]
Complex Numbers

- \( i^2 = -1 \)
- \( z = a + bi \), where \( a, b \in \mathbb{R} \)
- The complex conjugate of \( z \): \( \bar{z} = a - bi \)
- The length of \( z \) is \( |z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}} \)
- When \( |z| = 1 \), \( z = e^{i\theta} = \cos \theta + i \sin \theta \), where \( \theta \) is the “phase” of \( z \), i.e. the angle \( z \) forms with the positive real axis in the complex plane.
Complex linear algebra

- The inner product between complex vectors $|s\rangle = (s_1, s_2)$ and $|m\rangle = (m_1, m_2)$ is defined to be $\langle s|m\rangle = \bar{s}_1 m_1 + \bar{s}_2 m_2$. The bar indicates complex conjugation.
  - Now $\langle s| is no longer just the transpose of $|s\rangle$, but the complex conjugate of the transpose.
- The conjugate transpose (or adjoint) of a complex matrix $A$ is $A^\dagger$. $A^\dagger$ is obtained by first transposing $A$, and then taking a complex conjugate of each element.
- A matrix $U$ is **unitary** iff $UU^\dagger = U^\dagger U = I$
- Generalization for orthogonal matrices
Generalizing polarization

- A general polarization state of a photon can be represented by a normalized two-dimensional complex vector.
- Two such vectors $|s\rangle$ and $|t\rangle$ represent the same polarization state if there is a complex $\gamma$ such that $|s\rangle = \gamma |t\rangle$.
- Measurements are represented by complex orthonormal bases $(m^{(1)}, m^{(2)})$, where $|m^{(i)}\rangle = 0$ and $\langle m^{(1)}|m^{(2)}\rangle = 0$.
- All allowed reversible polarization transformations are represented by $2 \times 2$ unitary matrices.
Circular Polarization

- A complex vector $|s\rangle = (s_1 \ s_2)$ describes a circular polarization state iff $s_2 = \pm is_1$.
  - The $+$-case is called “right-hand circular polarization” and the $-$-case “left-hand circular polarization”. These are the only two circular polarization states.
- A circularly polarized photon has a 50% change to pass any polarizing filter, regardless of its orientation.
  - So the polarization doesn’t favor any particular axis. Hence the name “circular”.

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Photon Polarization
Elliptical Polarization

- A complex vector $|s\rangle = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$ describes an elliptical polarization iff $s_2/s_1 \neq \pm i$, but still has a nonzero imaginary part.
- Elliptically polarized photons do favor an axis over the other, but not as strongly as linearly polarized photons.
Summary

- Three different sorts of polarization: linear, circular and elliptical
- Polarization states can be represented by two-dimensional complex vectors.
- A polarization state can be “measured”
  - Measurements are represented by orthonormal bases
  - Makes photons probabilistically choose between two orthogonal states
  - The photons adopt the state they chose
- Polarization states can also be reversibly transformed
  - Allowed reversible transformations are represented by unitary matrices