Photon Polarization

T-79.4001 Seminar on Theoretical Computer Science

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Outline

Quantum Mechanics

- Developed between 1900 and 1926
- Radically changed our understanding of the physical world
  - Introduced probabilistic (undeterministic) behavior
  - It allows mutually exclusive situations to exist simultaneously in quantum superposition.
- We need to mathematically represent quantum
  - states
  - measurements
  - reversible transformations
  - (composite systems)
- My focus: photons and their polarization

Real Vectors 1/2

- **Definition**
  A column vector: $|s\rangle = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$
- **Definition**
  A row vector: $\langle s| = \begin{pmatrix} s_1 & s_2 \end{pmatrix}$
- **Definition**
  The dot/inner product between two real vectors $|s\rangle$ and $|m\rangle$:

$$\langle s|m\rangle = \begin{pmatrix} s_1 & s_2 \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = s_1 m_1 + s_2 m_2$$

Real Vectors 2/2

- **Definition**
  A vector is normalized if its length is 1.

- **Example**
  The following vectors are normalized: $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{pmatrix}$

- **Definition**
  Two vectors $|s\rangle$ and $|m\rangle$ are orthogonal (i.e. perpendicular) iff their inner product $\langle s|m\rangle$ is 0.

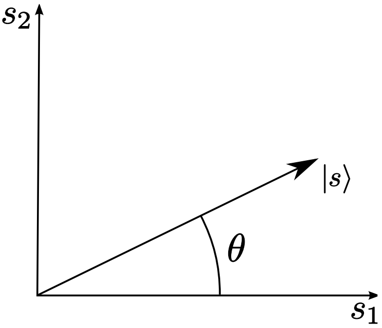
- **Example**
  The following vectors are mutually orthogonal:

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \cdot 0 + 0 \cdot 1 = 0$$

Representating Linear Polarization States

- A photon's linearization state is described by an axis on a plane.
- Such an axis is mathematically described by a normalized vector. The general form of a normalized vector is

$$\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$



- Vector $|s\rangle$ and its inverse $-|s\rangle$ represent the same linear polarization state.

Measurements

- There can be no device that directly ‘reads’ the polarization state of a photon.
- Instead, there are only devices that ‘ask’ photons binary questions. More accurately, such a device can have a photon choose between two mutually orthogonal states.
  - e.g. “Which are you: vertically or horizontally polarized?”
- Even if the photon is in neither state, it still has to make a choice between the two states. The choice is done in a probabilistic fashion.
- The photon’s state becomes the state that it chooses.

The Math of Measurements 1/2

- A polarization measurement device can be mathematically represented by an ordered pair of mutually orthogonal normalized state vectors (i.e. an orthonormal basis):

$$M = (|m^{(1)}\rangle, |m^{(2)}\rangle)$$

- When a photon with polarization state $|s\rangle$ is subjected to measurement M, it probabilistically chooses between $|m^{(1)}\rangle$ and $|m^{(2)}\rangle$. The probability of the photon choosing the state $|m^{(i)}\rangle$ is $|\langle s|m^{(i)}\rangle|^2$.
- So, the photon is much more likely to choose the state $|m^{(i)}\rangle$ that is closer to its original state $|s\rangle$.

The Math of Measurements 2/2

- Example (A measurement)

  Let
  $|m^{(1)}\rangle = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$, $|m^{(2)}\rangle = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$, and $|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
  What are the probabilities of the two outcomes?

$$p_1 = |\langle s|m^{(1)}\rangle|^2 = (1/2)^2 = 1/4$$
$$p_2 = |\langle s|m^{(2)}\rangle|^2 = (\sqrt{3}/2)^2 = 3/4$$

  - As expected $p_1 + p_2 = 1$.

Polarization Measurement Devices in Practise

- Wollaston prism
  - Takes a beam of unpolarized light and devides it into separate horozintally and vertically polarized beams.
- Polarizing filter
  - Like a wollaston prism, but polarizes part of the beam and simply absorbs the rest.

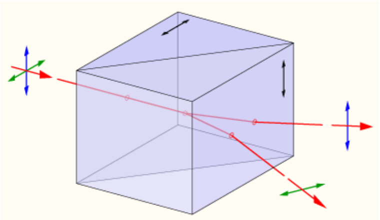


Figure: A Wollaston prism

Reversible Transformations

- A photon's polarization state can be reversibly transformed.
- Two types: (i) rotations and (ii) reflections around an axis
- The allowed transformations are mathematically represented by 2×2 real *orthogonal* matrices.
  - A matrix R is orthogonal iff $RR^T = R^TR = I$, where I is the identity matrix.
  - Let $|s\rangle$ be a polarization state and R be a transformation matrix. Then the outcome of transformation R on the state $|s\rangle$ is $R|s\rangle$.
- General rotation matrix: $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
- Reflection across a line of given angle

$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

Transformation examples

- **Example (A rotation)**

  Let's rotate a horizontal state vector $|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by 45° CCW:

$$\begin{pmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt(2) \\ 1/\sqrt(2) \end{pmatrix}$$

- **Example (A reflection)**

  Let's reflect the same vector $|s\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ around an axis with the angle 45°:

$$\begin{pmatrix} \cos(2 \cdot 45^\circ) & \sin(2 \cdot 45^\circ) \\ \sin(2 \cdot 45^\circ) & -\cos(2 \cdot 45^\circ) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Complex Numbers

- $i^2 = -1$
- $z = a + bi$, where $a, b \in \mathbb{R}$
- The complex conjugate of z: $\bar{z} = a - bi$
- The length of z is $|z| = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}}$
- When $|z| = 1$, $z = e^{i\theta} = \cos\theta + i\sin\theta$, where θ is the "phase" of z, i.e. the angle z forms with the positive real axis in the complex plane.

Complex linear algebra

- The inner product between complex vectors $|s\rangle = \binom{s_1}{s_2}$ and $|m\rangle = \binom{m_1}{m_2}$ is defined to be $\langle s|m\rangle = \bar{s}_1 m_1 + \bar{s}_2 m_2$. The bar indicates complex conjugation.
  - Now $\langle s|$ is no longer just the transpose of $|s\rangle$, but the complex conjugate of the transpose.
- The conjugate transpose (or adjoint) of a complex matrix A is $A^\dagger$. $A^\dagger$ is obtained by first transposing A, and then taking a complex conjugate of each element.
- A matrix U is unitary iff $UU^\dagger = U^\dagger U = I$
  - Generalization for orthogonal matrices

Generalizing polarization

- A general polarization state of a photon can be represented by a normalized two-dimensional complex vector.
- Two such vectors $|s\rangle$ and $|t\rangle$ represent the same polarization state if there is a complex γ such that $|s\rangle = \gamma|t\rangle$.
- Measurements are represented by complex orthonormal bases $(m^{(1)}, m^{(2)})$, where $|m^{(i)}| = 0$ and $\langle m^{(1)}|m^{(2)}\rangle = 0$.
- All allowed reversible polarization transformations are represented by 2×2 unitary matrices.

Circular Polarization

- A complex vector $|s\rangle = \binom{s_1}{s_2}$ describes a circular polarization state iff $s_2 = \pm i s_1$.
  - The $+$-case is called "right-hand circular polarization" and the $-$-case "left-hand circular polarization". These are the only two circular polarization states.
- A circularly polarized photon has a 50% change to pass any polarizing filter, regardless of its orientation.
  - So the polarization doesn't favor any particular axis. Hence the name "circular".

Elliptical Polarization

- A complex vector $|s\rangle = \binom{s_1}{s_2}$ describes an elliptical polarization iff $s_2/s_1 \neq \pm i$, but still has a nonzero imaginary part.
- Elliptically polarized photons do favor an axis over the other, but not as strongly as linearly polarized photons.

Summary

- Three different sorts of polarization: linear, circular and elliptical
- Polarization states can be represented by two-dimensional complex vectors.
- A polarization state can be "measured"
  - Measurements are represented by orthonormal bases
  - Makes photons probabilistically choose between two orthogonal states
  - The photons adopt the state they chose
- Polarization states can also be reversibly transformed
  - Allowed reversible transformations are represented by unitary matrices