Quantum Teleportation, Chapter 3.3
T-79.4001 Seminar on Theoretical Computer Science

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Outline

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2. Teleportation process
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Introduction

Teleportation process

Summary
No-Cloning theorem

- No-Cloning theorem crucial for preventing Eve from accessing bits sent by Alice
- Causes problems for Alice and Bob too:
  - Photon detectors are not perfect and will occasionally produce false detections.
  - Photon passing through channel like optical fiber has some probability of being degraded or destroyed.
  - So even if a small percentage of Alice’s photons manages to make through the long fiber, it gets impossible for Bob to distinguish Alice’s photons from the background noise.
  - If photons could be cloned, one could place cloning devices at regular intervals along the fibre but this is not possible.
One solution to the problem is quantum teleportation.

If we can arrange two perfectly entangled particles, one at location A and other at B, a state of a third photon arriving at A can be teleported directly to B.

By repeating this process, it is possible in principle to extend key distribution scheme to arbitrary distance.

Though arranging a well-separated entangled pair is difficult.
Bell measurement

Four mutually orthogonal two-qubit states, each of which is maximally entangled:

\[ |\Phi^+\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle) \]
\[ |\Phi^-\rangle = (1/\sqrt{2})(|00\rangle - |11\rangle) \]
\[ |\Psi^+\rangle = (1/\sqrt{2})(|01\rangle + |10\rangle) \]
\[ |\Psi^-\rangle = (1/\sqrt{2})(|01\rangle - |10\rangle) \]

This is called Bell measurement, after John S. Bell
Let us assume that we have particles a and b at locations A (Alice) and B (Bob) and they are in entangled state $|\Phi^+\rangle$.

At some point particle c arrives at location A whose state $|s\rangle = \alpha |0\rangle + \beta |1\rangle$ is to be teleported to B.

The combined state of the particles c, a and b, where cabs is the order in each term, can be written as:

$$|s\rangle \otimes |\Phi^+\rangle = \frac{1}{\sqrt{2}} \left( \alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle \right)$$
Teleportation, step 1

- Alice performs a Bell measurement to subsystem consisting of particles a and c.
- To obtain result of this measurement we use rule for measurements on subsystems defined on previous lecture and reexpress the above state in terms of the outcome vectors of the measurement.
- No physical change is done yet.
Alice makes the Bell measurement on particles a and c. Above equation shows us the state of the particle b after the measurement.

For example, if Alice gets the outcome $|\Phi^+\rangle$, the state of the particle b is $\alpha|0\rangle + \beta|1\rangle$. 

```latex
\begin{align*}
|s\rangle \otimes |\Phi^+\rangle &= \frac{1}{2} \left[ |\Phi^+\rangle \otimes \left( \alpha|0\rangle + \beta|1\rangle \right) \\
&\quad + |\Phi^−\rangle \otimes \left( \alpha|0\rangle - \beta|1\rangle \right) \\
&\quad + |\Psi^+\rangle \otimes \left( \beta|0\rangle + \alpha|1\rangle \right) \\
&\quad + |\Psi^−\rangle \otimes \left( -\beta|0\rangle + \alpha|1\rangle \right) \right]
\end{align*}
```
Alice sends to Bob the outcome of her measurement through a classical channel.

Bob performs a unitary transformation to particle b. Transformation depends on the result of Alice’s measurement:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Transformation</th>
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<tbody>
<tr>
<td>$</td>
<td>\Phi^+\rangle$</td>
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<td>\Phi^-\rangle$</td>
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<tr>
<td>$</td>
<td>\psi^+\rangle$</td>
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<tr>
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<td>\psi^-\rangle$</td>
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\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
Once Bob has performed the appropriate transformation, particle b is guaranteed to be in the state $|s\rangle$.

Example: Alice gets the outcome $|\psi^+\rangle$ so Bob’s particle is in state $\beta|0\rangle + \alpha|1\rangle$. Bob applies the transformation $X$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = |s\rangle$$
State $|s\rangle$ was not cloned, the states of the a and c particles were collapsed in the measurement.

Classical communication from Alice to Bob contains no information about the state of $|s\rangle$.

Probabilities of the four outcomes of Alice’s measurement are all equal to $1/4$ regardless of values $\alpha$ and $\beta$.

For example, probability of outcome $|\Phi^+\rangle$:

The probability of outcome $|m_i\rangle$ is $p_i = \langle v_i | v_i \rangle$

$$\alpha^2 + \beta^2 = 1 \Rightarrow ((1/2)\alpha)^2 + ((1/2)\beta)^2 = (1/4)\alpha^2 + (1/4)(1-\alpha^2) = 1/4$$
Teleportation, observations

- Teleportation is not instantaneous. Bob needs the signal from Alice to tell him the outcome of her measurement. Without it he has nothing useful.
Creating a perfectly entangled pair is difficult.

Optical fibers are imperfect. The pair will not be anymore perfectly entangled when the other member has been sent over the fiber.

Entanglement purification. Can be done in principle as long as the channel is not too noisy.

So again the imperfectness of the fiber sets us limitations on transmission distance.
However we may chain multiple entangled pairs spanning some shorter distance to achieve arbitrary transmission distances.

And probably use Bennett-Brassard key distribution scheme to prevent Eve from accessing any of the "middle links".

Currently (book published in 2006) we don’t have the technology to do entanglement purification on large scale.
In principle, quantum teleportation can be used to extend quantum key distribution scheme to arbitrary distances. Some classical channel is still needed for signaling the outcome of the measurement and probably also for arranging a perfectly entangled pair. Teleportation has been successfully demonstrated in a number of experiments but entanglement purification is not yet possible on a large scale.