T-79.4001 Seminar on Theoretical Computer Science
Protecting Information –
General quantum variables and composite systems
Chapter 2.2, 2.3

Tommi Häkkinen

6.2.2008

What is quantum variable?

- Examples of quantum variables:

Variable	N
Photon polarization	2
Photon momentum	∞
Electron spin	2
Electron momentum	∞
Quark spin	2
Quark color	3

- The maximum number of perfectly distinguishable states N (dimension).
- If two variables have the same N, then their quantum descriptions are the same.
- Special importance of variables with N=2. Generic word “qubit”.

Quantum variables - Qubit

- The electron spin and photon polarization are two examples of qubits.
- Similarities with a classical bit – 0, 1 or superposition of both.
- Qubit can be represented as a linear combination of $|0>$ and $|1>$:

  $$|\psi = \alpha|0> + \beta|1>$$

  where α and β are probability amplitudes.
- It follows that α and β must be constrained by the equation:

  $$|\alpha|^2+|\beta|^2=1$$

- Quantum computer – consists of many qubits.

Quantum variables – Basic quantum rules

Definition 1:

- Let $|s\rangle = \begin{pmatrix} s1 \\ \dots \\ s_N \end{pmatrix}$ and $|m\rangle = \begin{pmatrix} m1 \\ \dots \\ m_N \end{pmatrix}$ be complex N-dimensional vectors.
- The inner product $\langle s|m\rangle$ is the complex number $\overline{s_1} m_1 + \ldots + \overline{s_N} m_N$ (state space).
- The vectors $|s\rangle$ and $|s\rangle$ are orthogonal iff $\langle s|m\rangle = 0$.

Definition 2:

- The length of the vector $|s\rangle$ is $\langle s|s\rangle^{1/2}$.
- If the length is 1, it is said to be normalized (state vector).

Quantum variables – Basic quantum rules

Rule 1:

- Two vectors $|s>$ and $|t>$ represent the same state iff they are complex scalar multiples of each other.
- Since both vectors are normalized, the scalar factor must be of the form $e^{i\phi}$; that is, unit magnitude.

Rule 2:

- A standard measurement is presented by an orthonormal basis $(|m^{(1)}>, \ldots, |m^{(N)}>)$.
- Every basis represents a possible measurement.
- Each possible outcome of the measurement is associated with one and only one of the vectors. Probability of the outcome $|m^{(i)}>$ is $|<s|m^{(i)}>|^2$ when $|s>$ is the initial state.

Quantum variables - Basic quantum rules

Rule 3:

- Every allowed reversible physical transformation on the state is represented by an N x N unitary matrix U, and every such U represents an allowed transformation.

- About Rule 2. Possible outcomes of a standard measurements are mutually exclusive. Therefore probabilities adds up to 1.

Theorem:

- Let $|s>$ be normalized N-dimensional vector and let $M= (|\mathrm{m}^{(1)}>, \ldots, |\mathrm{m}^{(N)}>)$ be an orthonormal basis.
- Let $p_i = |<s|m^{(i)}>|^2$, $i=1,\ldots N$. Then $p_i + \ldots + p_N = 1$.

Quantum variables - Basic quantum rules

Example 1:

- The unitary matrix $U=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ represents an allowed transformation of the spin state of an electron.
- Apply the transformation to an electron in the "up" state $\begin{pmatrix}1\\ 0\end{pmatrix}$. The spin state after the transformation: $|s'>=U|s>=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}=\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$.
- The probability that the outcome of up-vs-down measurement on the electron is "up": $p_{\uparrow}=|\langle\uparrow|s'\rangle|^2=1/2$

Quantum variables - Basic quantum rules

Example 2:

- The state of a certain meson is represented by the vector $|s'>= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.
- Find a measurement M such that when it is performed on this particle, the three outcomes are equally likely.
- Three orthonormal vectors $\{|m^{(1)}>, |m^{(2)}>, |m^{(3)}>\}$ for which their first component is $1/\sqrt{(3)}$. Here is one such set:

$$M= \left\{ 1/\sqrt{(3)} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 1/\sqrt{(3)} \begin{pmatrix} 1 \\ \omega \\ \overline{\omega} \end{pmatrix}, 1/\sqrt{(3)} \begin{pmatrix} 1 \\ \overline{\omega} \\ \omega \end{pmatrix} \right\} \quad \text{where} \quad \omega = e^{2\pi i/3}$$

Composite Systems

- When thinking of quantum computer, which might consist of thousands of qubits, then quantum mechanics of composite systems needs to be known.
- In classical physics the transition from a single object to a system of many objects is trivial, but in quantum mechanics the situation is different.
- For example, two photons can be in different galaxies.
- Basis states for a single photon |↔> and |↕>.
- Basis states for a pair of photons |↔↔>, |↔↕>, |↕↔> and |↕↕>.
- The most general polarization state of a pair of photons (linear combination):

$$|s>=s1|\leftrightarrow\leftrightarrow>+s2\ |\leftrightarrow\updownarrow>+s3|\updownarrow\leftrightarrow>+s4|\updownarrow\updownarrow>$$

Composite Systems – Tensor product of two vector spaces

Definition :

- Let $\{|b^{(1)}>, \ldots, |b^{(N)}>\}$ and $\{|c^{(1)}>, \ldots, |c^{(M)}>\}$ be orthonormal bases for two complex vector spaces H_N and H_M .
- Tensor product $H_N \otimes H_M$:

  New set of basis vectors $\{|b^{(i)} c^{(j)}>\}$ *, i=1,...N, j=1...M,* which are orthonormal.

  Tensor product consists of all complex linear combinations

  $\sum_{ij} s_{ij} |b^{(i)} c^{(j)}>$

  If $|s>= \sum_{ij} s_{ij} |b^{(i)} c^{(j)}>$ and $|t>= \sum_{ij} t_{ij} |b^{(i)} c^{(j)}>$ then $<s|t>= \sum_{ij} \overline{s_{ij}}\, t_{ij}$

Composite Systems – Tensor product of two vectors

Definition :

- Let $|v\rangle = \sum_i v_i |b^{(i)}\rangle$ be an element of H_N and $|w\rangle = \sum_j w_j |c^{(j)}\rangle$ be an element of H_M
- Tensor product of $|v\rangle$ with $|w\rangle$: $\sum_{ij} v_i w_j |b^{(i)} c^{(j)}\rangle$

Example:

- $|s\rangle = \begin{pmatrix} s1 \\ s2 \end{pmatrix} = s1|\leftrightarrow\rangle + s2\ |\updownarrow\rangle$ is the polarization state of photon a and
  $|t\rangle = \begin{pmatrix} t1 \\ t2 \end{pmatrix} = t1|\leftrightarrow\rangle + t2\ |\updownarrow\rangle$ is the polarization state of photon b.
- The polarization state of the pair ab is: $|s\rangle \otimes |t\rangle = (s1|\leftrightarrow\rangle + s2\ |\updownarrow\rangle) \otimes (t1|\leftrightarrow\rangle + t2|\updownarrow\rangle)$
  $= s1t1|\leftrightarrow\leftrightarrow\rangle + s1t2|\leftrightarrow\updownarrow\rangle + s2t1|\leftrightarrow\updownarrow\rangle + s2t2|\updownarrow\updownarrow\rangle = \begin{pmatrix} s1t1 \\ s1t2 \\ s2t1 \\ s2t2 \end{pmatrix}$

Composite Systems – Tensor product of two operators

Definition :

- Skip the definition.

Example:

- Given two matrices $T = \begin{pmatrix} a\,b \\ c\,d \end{pmatrix}$ and $T = \begin{pmatrix} w\,x \\ y\,z \end{pmatrix}$, then their tensor product

$$T \otimes V = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

Composite Systems – The composite system rule

- Let H_N and H_M be the state spaces of two quantum variables A and B (e.g. photon polarizations).
- The allowed states of the combined system AB are represented by the normalized vectors in $H_N \otimes H_M$.
- The combined system AB follows all the rules of quantum mechanics for a variable with NM dimensions.

Composite Systems – Product state

- $|s>$ is a state of A and $|t>$ is a state of B. Then $|s>\otimes|t>$ is a possible state of AB; that is product state.

Example 1:

- Photon 1 is polarized at an angle 45° and photon 2 is horizontally polarized.
- Polarization state of the pair:

  $$t>=[\frac{1}{\sqrt{(2)}}(|\leftrightarrow> +|\updownarrow>)] \otimes |\leftrightarrow>= \frac{1}{\sqrt{(2)}}(|\leftrightarrow\leftrightarrow>+|\updownarrow\leftrightarrow>).$$

- Both photons have state of their own so the joint state is a product state.

Composite Systems – Product state

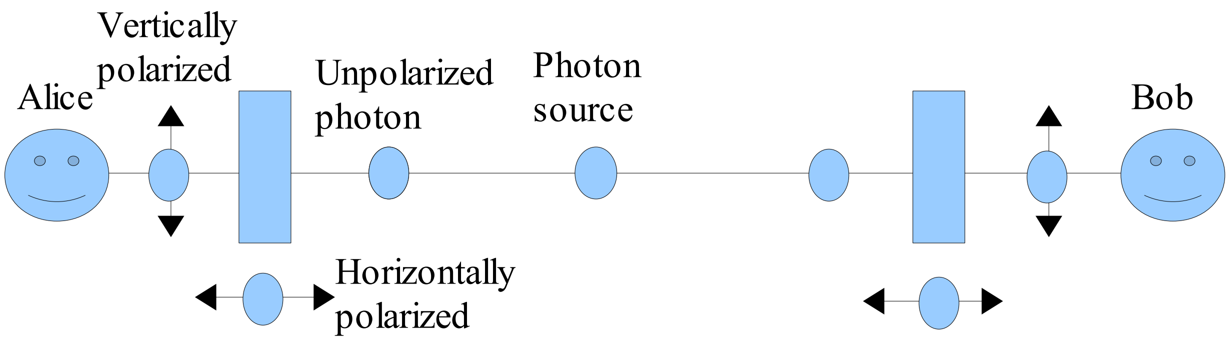
Example 2:

- Polarization state of a pair of photons: $|s\rangle = \frac{1}{\sqrt{(2)}} (|\leftrightarrow\leftrightarrow\rangle + |\updownarrow\updownarrow\rangle) = \begin{pmatrix} 1/\sqrt{(2)} \\ 0 \\ 0 \\ 1/\sqrt{(2)} \end{pmatrix}$.
- Is not a product state.
- By definition, a product state is of the form:

  $|t\rangle = (a|\leftrightarrow\rangle + b|\updownarrow\rangle) \otimes (c|\leftrightarrow\rangle + d|\updownarrow\rangle)$
  $= ac|\leftrightarrow\leftrightarrow\rangle + ad|\leftrightarrow\updownarrow\rangle + bc|\updownarrow\leftrightarrow\rangle + bd|\updownarrow\updownarrow\rangle.$
- In the state $|s\rangle$ the coefficient of $|\leftrightarrow\updownarrow\rangle$ is zero so either a or d must be zero. But then $|\leftrightarrow\leftrightarrow\rangle$ or $|\updownarrow\updownarrow\rangle$ would also have to be zero, which is not the case.

Composite Systems – Entangled state

- Any state of any composite system that is not a product state is called an entangled state.
- It is possible to prepare e.g. two photons in a single quantum state such that each photon have polarization orthogonal to the other.
- If two people each receive one of the entangled photons, they will find that the other person's photon has orthogonal polarization (no matter how far apart they are).



Composite Systems – Entangled state

- When Alice measures the polarization of her photon, we instantly know the polarization of Bob's photon also.
- Quantum entanglement does not enable the transmission of classical informaiton faster than light.
- Alice's measurement is random and she can't decide which state to collapse the composite system – can't transfer information – causality is preserved (See EPR paradox).