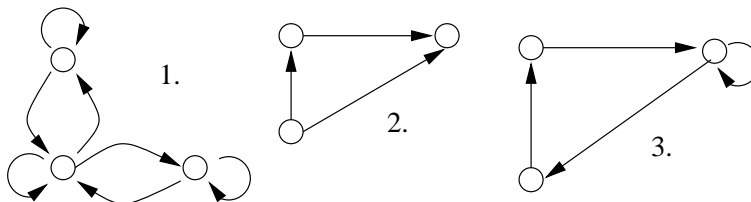


T-79.3001 Logic in Computer Science: Foundations **Spring 2009**
Exercise 8 ([Nerode and Shore, 1997], Predicate Logic, Chapters 4 and 9)
March 26 – March 30, 2009

Solutions to demonstration problems

Solution to Problem 4

The graphs given below illustrate different properties of relations. Here the nodes are the elements in a structure and there is an edge between two nodes $x \in A, y \in A$ if and only if $R(x, y)$ is true for x, y .



Reflexivity ($\forall x R(x, x)$) means that every node in the graph has an edge to itself and irreflexivity ($\forall x \neg R(x, x)$) means that no node has an edge to itself. First of the graphs is reflexive, the second irreflexive and the third is neither reflexive nor irreflexive.

Symmetry ($\forall x \forall y (R(x, y) \rightarrow R(y, x))$) means that whenever there is an edge from x to y , there is also an edge from y to x . Asymmetric ($\forall x \forall y (R(x, y) \rightarrow \neg R(y, x))$) graph has no edge from y to x if there is edge from x to y . The first graph is symmetric, the second asymmetric and the third is neither.

In a transitive graph ($\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$) if there is a path from x to y along the edges, then there is an edge from x to y in the graph. The second graph is transitive.

In a serial graph ($\forall x \exists y R(x, y)$) there is at least one edge from each node x . The first and the third graph are serial.

Now define relations $T(x, y)$ (x knows y), $N(x, y)$ (x is married to y), $V(x, y)$ (y is a parent of x) ja $E(x, y)$ (y is an ancestor of x). These relations have the following properties.

Relation	refl.	irrefl.	symm.	asymm.	trans.	serial.
knows	*		*			*
married to		*	*			
parent		*		*		*
ancestor		*		*	*	*

Solution to Problem 5

- a) Consider \mathcal{S} with domain $U = \{1, 2\}$ and $P^{\mathcal{S}} = \{\langle 1, 1 \rangle, \langle 2, 2 \rangle\}$. Now it holds $\mathcal{S} \models \forall x \exists y P(x, y)$ and $\mathcal{S} \not\models \exists y \forall x P(x, y)$ (there is no value for y such that for all x we would have $\langle x, y \rangle \in P^{\mathcal{S}}$). Thus the implication is false in \mathcal{S} .
- b) Consider \mathcal{S} with domain $U = \{1\}$ and $P^{\mathcal{S}} = \{1\}, Q^{\mathcal{S}} = \emptyset$. Now the left side of the implication is true and the right side false in \mathcal{S} , and \mathcal{S} is a counterexample.
- c) Consider \mathcal{S} with domain $U = \{1\}$ ja $P^{\mathcal{S}} = \emptyset, R^{\mathcal{S}} = \{1\}$. Now $\forall x (P(x) \rightarrow R(x))$ is true in \mathcal{S} since the left side of the implication is false in \mathcal{S} . Similarly $\mathcal{S} \models \forall x (P(x) \rightarrow \neg R(x))$.

Solution to Problem 6

- Remove connectives \rightarrow and \leftrightarrow .
- Negations in, quantifiers out.
- Use distribution rules to obtain CNF / DNF.

a)

$$\begin{aligned}
 & \forall y (\exists x P(x, y) \rightarrow \forall z Q(y, z)) \wedge \exists y (\forall x R(x, y) \vee \forall x Q(x, y)) \\
 & \equiv \forall y (\neg \exists x P(x, y) \vee \forall z Q(y, z)) \wedge \exists y (\forall x R(x, y) \vee \forall x Q(x, y)) \\
 & \equiv \forall y (\forall x \neg P(x, y) \vee \forall z Q(y, z)) \wedge \exists y (\forall x R(x, y) \vee \forall x Q(x, y)) \\
 & \equiv \exists y_1 (\forall y (\forall x \neg P(x, y) \vee \forall z Q(y, z)) \wedge (\forall x R(x, y_1) \vee \forall x Q(x, y_1))) \\
 & \equiv \exists y_1 \forall y_2 ((\forall x \neg P(x, y_2) \vee \forall z Q(y_2, z)) \wedge (\forall x R(x, y_1) \vee \forall x Q(x, y_1))) \\
 & \equiv \exists y_1 \forall y_2 \forall x_1 \forall x_2 \forall z \forall x_3 ((\neg P(x_1, y_2) \vee Q(y_2, z)) \wedge (R(x_2, y_1) \vee Q(x_3, y_1)))
 \end{aligned}$$

This is the Prenex normal form and the part inside quantifiers is in CNF.
Skolemization:

$$\forall y_2 \forall x_1 \forall x_2 \forall z \forall x_3 ((\neg P(x_1, y_2) \vee Q(y_2, z)) \wedge (R(x_2, c) \vee Q(x_3, c)))$$

c)

$$\begin{aligned}
& \forall x \exists y Q(x, y) \vee (\exists x \forall y P(x, y) \wedge \neg \exists x \exists y P(x, y)) \\
& \equiv \forall x \exists y Q(x, y) \vee (\exists x \forall y P(x, y) \wedge \forall x \forall y \neg P(x, y)) \\
& \equiv \forall x \exists y Q(x, y) \vee \exists x_1 \forall y_1 \forall x_2 \forall y_2 (P(x_1, y_1) \wedge \neg P(x_2, y_2)) \\
& \equiv \exists x_1 \forall x_3 \exists y_3 \forall y_1 \forall x_2 \forall y_2 (Q(x_3, y_3) \vee (P(x_1, y_1) \wedge \neg P(x_2, y_2)))
\end{aligned}$$

This is the Prenex normal form and we continue to get CNF.

$$\exists x_1 \forall x_3 \exists y_3 \forall y_1 \forall x_2 \forall y_2 ((Q(x_3, y_3) \vee P(x_1, y_1)) \wedge (Q(x_3, y_3) \vee \neg P(x_2, y_2)))$$

Skolemization:

$$\forall x_3 \forall y_1 \forall x_2 \forall y_2 ((Q(x_3, f(x_3)) \vee P(c, y_1)) \wedge (Q(x_3, f(x_3)) \vee \neg P(x_2, y_2)))$$

Solution to Problem 7

a)

$$\begin{aligned}
& \forall x \phi(x) \rightarrow \psi \\
& \equiv \neg \forall x \phi(x) \vee \psi \\
& \equiv \exists x \neg \phi(x) \vee \psi \\
& \equiv \exists x_1 (\neg \phi(x_1) \vee \psi) \\
& \equiv \exists x_1 (\phi(x_1) \rightarrow \psi)
\end{aligned}$$

b) Similarly, $\exists x \phi(x) \rightarrow \psi \equiv \forall x_1 (\phi(x_1) \rightarrow \psi)$.

c)

$$\begin{aligned}
& \phi \rightarrow \forall x \psi(x) \\
& \equiv \neg \phi \vee \forall x \psi(x) \\
& \equiv \forall x_1 (\neg \phi \vee \psi(x_1)) \\
& \equiv \forall x_1 (\phi \rightarrow \psi(x_1))
\end{aligned}$$

d) Similarly, $\phi \rightarrow \exists x \psi(x) \equiv \exists x_1 (\phi \rightarrow \psi(x_1))$.

Solution to Problem 8

- a) Sentence $\neg \exists x((P(x) \rightarrow P(a)) \wedge (P(x) \rightarrow P(b)))$:
Eliminate implications: $\neg \exists x((\neg P(x) \vee P(a)) \wedge (\neg P(x) \vee P(b)))$.
Push \neg inside $\exists x$:
 $\forall x \neg((\neg P(x) \vee P(a)) \wedge (\neg P(x) \vee P(b)))$.
Push negations inside the formula:
 $\forall x((P(x) \wedge \neg P(a)) \vee (P(x) \wedge \neg P(b)))$.
Bring $P(x)$ outside: $\forall x(P(x) \wedge (\neg P(a) \vee \neg P(b)))$.
Drop universal quantifiers: $P(x) \wedge (\neg P(a) \vee \neg P(b))$.
Clausal form: $\{\{P(x)\}, \{\neg P(a), \neg P(b)\}\}$.
- b) Sentence $\forall y \exists x P(x, y)$:
Skolemization: $\forall y P(f(y), y)$.
Drop universal quantifiers: $P(f(y), y)$.
Clausal form: $\{\{P(f(y), y)\}\}$.
- c) Sentence $\neg \forall y \exists x G(x, y)$:
Push \neg inside $\forall y$: $\exists y \neg \exists x G(x, y)$.
Push \neg inside $\exists x$: $\exists y \forall x \neg G(x, y)$.
Skolemization: $\forall x \neg G(x, c)$.
Drop universal quantifiers: $\neg G(x, c)$.
Clausal form: $\{\{\neg G(x, c)\}\}$.
- d) Sentence $\exists x \forall y \exists z (P(x, z) \vee P(z, y) \rightarrow G(x, y))$:
Eliminate implication: $\exists x \forall y \exists z (\neg (P(x, z) \vee P(z, y)) \vee G(x, y))$.
Push negations inside:
 $\exists x \forall y \exists z ((\neg P(x, z) \wedge \neg P(z, y)) \vee G(x, y))$.
Push $G(x, y)$ inside the formula:
 $\exists x \forall y \exists z ((\neg P(x, z) \vee G(x, y)) \wedge (\neg P(z, y) \vee G(x, y)))$.
Skolemization: $\forall y \exists z ((\neg P(c, z) \vee G(c, y)) \wedge (\neg P(z, y) \vee G(c, y)))$.
Skolemization: $\forall y ((\neg P(c, f(y)) \vee G(c, y)) \wedge (\neg P(f(y), y) \vee G(c, y)))$.
Drop universal quantifiers:
 $(\neg P(c, f(y)) \vee G(c, y)) \wedge (\neg P(f(y), y) \vee G(c, y))$.
Clausal form:
 $\{\{\neg P(c, f(y)), G(c, y)\}, \{\neg P(f(y), y), G(c, y)\}\}$.