T-79.3001 Logic in Computer Science: Foundations **Spring 2009**
Exercise 11 ([Nerode and Shore, 1997], Predicate Logic, Chapters 10 – 14)
May 5 – May 5, 2009

Solutions to demonstration problems

4. Define the Herbrand universe and Herbrand base for the following sets of clauses.

a) $\{\{\neg G(x,c)\}\}$,

b) $\{\{P(f(y),y)\}\}$,

c) $\{\{P(x)\},\ \{\neg P(a),\neg P(b)\}\}$,

d) $\{\{\neg P(x,y),\neg P(y,z),G(x,z)\}\}$,

e) $\{\{\neg P(x,y)\},\ \{Q(a,x),Q(b,f(y))\}\}$, ja

f) $\{\{P(x),Q(f(x,y))\}\}$

Solution to Problem 4

a) $U=\{c\}$, $B=\{G(c,c)\}$.

b) $U=\{a,f(a),f(f(a)),\ldots\}$, $B=\{P(e_1,e_2)\,|\,e_1\in U,e_2\in U\}$.

c) $U=\{a,b\}$, $B=\{P(a),P(b)\}$.

d) $U=\{a\}$, $B=\{P(a,a),G(a,a)\}$.

e) $U=\{a,b,f(a),f(b),f(f(a)),f(f(b)),\ldots\}$,
$B=\{P(e_1,e_2)\,|\,e_1\in U,e_2\in U\}\cup\{Q(e_1,e_2)\,|\,e_1\in U,e_2\in U\}$.

f) $U=\{a,f(a,a),f(a,f(a,a)),f(f(a,a),a),f(f(a,a),f(a,a)),\ldots\}$,
$B=\{P(e)\,|\,e\in U\}\cup\{Q(e)\,|\,e\in U\}$.

5. Consider

$$\Sigma=\{\forall x P(x,a,x),\ \neg\exists x\exists y\exists z(P(x,y,z)\wedge\neg P(x,f(y),f(z)))\}.$$

a) Transform Σ into a set of clauses S.

b) Define the Herbrand universe H and Herbrand base B of S.

c) Let Herbrand structures be subsets of the Herbrand base. Find the subset minimal and maximal Herbrand models of S.

Solution to Problem 5

a) A clause $\{P(x,a,x)\}$ is obtained from the sentence $\forall x P(x,a,x)$. and the other sentence $\neg(\exists x \exists y \exists z (P(x,y,z) \wedge \neg P(x, f(y), f(z)))$ results in clause
$\{\neg P(x,y,z), P(x, f(y), f(z))\}$. Thus we get

$$S = \{\{P(x,a,x)\}, \{\neg P(x,y,z), P(x, f(y), f(z))\}\}.$$

b) Herbrand-universe $H = \{a, f(a), f(f(a)), \ldots\} = \{f^n(a) \mid n \geq 0\}$ and Herbrand-base $B = \{P(e_1, e_2, e_3) \mid e_1, e_2, e_3 \in H\}$.

c) The maximal Herbrand-model for S is B, since every term of the form $P(f^n(a), a, f^n(a))$, $n \geq 0$ belongs to B (the first clause is satisfied), and each term of the form $P(f^n(a), f^{m+1}(a), f^{k+1}(a))$, for $n, m, k \geq 0$, belongs to B (the second clause is satisfied).

The minimal Herbrand-model is $\{P(a,a,a), P(a, f(a), f(a))\}$.

6. Transform the problem of deciding the validity of sentence

$$\exists x \exists y (P(x) \leftrightarrow \neg P(y)) \rightarrow \exists x \exists y (\neg P(x) \wedge P(y))$$

into the problem of satisfiability of a propositional logic statement and solve the problem.

Solution to Problem 6

Find the set of clauses S which is the clausal form of the sentence (finite, contains no function symbols), find the Herbrand universe H of S and furthermore, the finite set of Herbrand-instances S'. This can be interpreted as a set of propositional clauses and for instance resolution can be used to check the validity of S'.

7. Find the composition of substitutions $\{x/y, y/b, z/f(x)\}$ and $\{x/g(a), y/x, w/c\}$.

Solution to Problem 7

$$\{y/b, z/f(g(a)), w/c\}$$

8. Find the most general unifiers for the following sets of literals.

a) $\{P(x, g(y), f(a)), P(f(y), g(f(z)), z)\}$

b) $\{P(x, f(x), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$

c) $\{P(x, f(x, y)), P(y, f(y, a)), P(b, f(b, a))\}$

d) $\{P(f(a), y, z), P(y, f(a), b), P(x, y, f(z))\}$

Solution to Problem 8

a) $\sigma_0 = \varepsilon$ (empty substitution)
$S_0 = \{P(x, g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_0) = \{x, f(y)\}$
$\sigma_1 = \{x/f(y)\}$
$\sigma_0\sigma_1 = \{x/f(y)\}$
$S_1 = \{P(f(y), g(y), f(a)), P(f(y), g(f(z)), z)\}$
$D(S_1) = \{y, f(z)\}$
$\sigma_2 = \{y/f(z)\}$
$\sigma_0\sigma_1\sigma_2 = \{x/f(f(z)), y/f(z)\}$
$S_2 = \{P(f(f(z)), g(f(z)), f(a)), P(f(f(z)), g(f(z)), z)\}$
$D(S_2) = \{f(a), z\}$
$\sigma_3 = \{z/f(a)\}$

$\sigma_0\sigma_1\sigma_2\sigma_3 = \{x/f(f(f(a))), y/f(f(a)), z/f(a)\}$
$S_3 = \{P(f(f(f(a))), g(f(f(a))), f(a))\}$
MGU is $\sigma_0\sigma_1\sigma_2\sigma_3$.

b) $\sigma_0 = \varepsilon$
$S_0 = \{P(x, f(x), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_0) = \{x, a, y\}$
$\sigma_1 = \{x/a\}$
$S_1 = \{P(a, f(a), g(y)), P(a, f(g(a)), g(a)), P(y, f(y), g(a))\}$
$D(S_1) = \{a, y\}$
$\sigma_2 = \{y/a\}$
$S_2 = \{P(a, f(a), g(a)), P(a, f(g(a)), g(a))\}$
$D(S_2) = \{a, g(a)\}$
Terms a and $g(a)$ cannot be unified.

c) $\sigma_0 = \varepsilon$
$S_0 = \{P(x, f(x,y)), P(y, f(y,a)), P(b, f(b,a))\}$
$D(S_0) = \{x, y, b\}$
$\sigma_1 = \{x/b\}$
$S_1 = \{P(b, f(b,y)), P(y, f(y,a)), P(b, f(b,a))\}$
$D(S_1) = \{b, y\}$
$\sigma_2 = \{y/b\}$
$S_2 = \{P(b, f(b,b)), P(b, f(b,a))\}$
$D(S_2) = \{b, a\}$
Terms b and a cannot be unified.

d) $\sigma_0 = \varepsilon$
$S_0 = \{P(f(a), y, z), P(y, f(a), b), P(x, y, f(z))\}$
$D(S_0) = \{f(a), y, x\}$
$\sigma_1 = \{y/f(a)\}$
$S_1 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(x, f(a), f(z))\}$
$D(S_1) = \{f(a), x\}$
$\sigma_2 = \{x/f(a)\}$
$S_2 = \{P(f(a), f(a), z), P(f(a), f(a), b), P(f(a), f(a), f(z))\}$
$D(S_2) = \{z, b, f(z)\}$
$\sigma_3 = \{z/b\}$
$S_3 = \{P(f(a), f(a), b), P(f(a), f(a), f(b))\}$
$D(S_3) = \{b, f(b)\}$
Terms b and $f(b)$ cannot be unified.

9. Show that

a) the composition of substitutions is not commutative, that is, there are substitutions σ and λ such that $\sigma\lambda \neq \lambda\sigma$.

b) a most general unifier is not unique, that is, there is a set of literals S such that it has two most general unifiers σ and λ such that $\sigma \neq \lambda$.

Solution to Problem 9

a) Consider $\sigma = \{x/a\}$ and $\lambda = \{x/b\}$. Now, $\sigma\lambda \neq \lambda\sigma$.

b) $S = \{P(x), P(y)\}$ has two MGUs: $\{x/y\}$ and $\{y/x\}$.

10. Unify $\{P(x,y,z), P(f(w,w), f(x,x), f(y,y))\}$.

Solution to Problem 10

$$\{x/f(w,w), y/f(f(w,w), f(w,w)),$$
$$z/f(f(f(w,w), f(w,w)), f(f(w,w), f(w,w)))\}.$$

11. Use resolution to prove that there are no barbers, when

a) all barbers shave everyone who does not shave himself, and

b) no barber shaves anyone who shaves himself.

Solution to Problem 11

Define $P(x)$ = "x is barber" and $A(x,y)$ = "x shaves y".

a) $\forall x(P(x) \rightarrow \forall y(\neg A(y,y) \rightarrow A(x,y)))$,

b) $\forall x(P(x) \rightarrow \forall y(A(y,y) \rightarrow \neg A(x,y)))$.

The clausal form:

a) $\forall x(P(x) \rightarrow \forall y(\neg A(y,y) \rightarrow A(x,y)))$
$\forall x(\neg P(x) \vee \forall y(A(y,y) \vee A(x,y)))$
$\forall x \forall y(\neg P(x) \vee A(y,y) \vee A(x,y))$
$\neg P(x) \vee A(y,y) \vee A(x,y)$
$\{\neg P(x_1),\ A(y_1,y_1),\ A(x_1,y_1)\}$

b) $\forall x(P(x) \to \forall y(A(y,y) \to \neg A(x,y)))$
$\forall x(\neg P(x) \vee \forall y(\neg A(y,y) \vee \neg A(x,y)))$
$\forall x \forall y(\neg P(x) \vee \neg A(y,y) \vee \neg A(x,y))$
$\neg P(x) \vee \neg A(y,y) \vee \neg A(x,y)$
$\{\neg P(x_2),\ \neg A(y_2,y_2),\ \neg A(x_2,y_2)\}$

We want to show $\neg\exists x P(x)$, and thus consider its negation $\exists x P(x)$. In the clausal form: $\{P(a)\}$.

From clauses

$$\{\neg P(x_1), A(y_1,y_1), A(x_1,y_1)\} \qquad \text{and} \qquad \{\neg P(x_2),\ \neg A(y_2,y_2),\ \neg A(x_2,y_2)\}$$

we get

$$\{\neg P(x_3)\} \qquad (\text{substitution } \{x_1/x_3, x_2/x_3, y_1/x_3, y_2/x_3\})$$

From clauses $\{P(a)\}$ and $\{\neg P(x_3)\}$ we obtain the empty clause (substitution $\{x_3/a\}$). Thus the set of clauses is unsatisfiable and $\neg\exists x P(x)$ is a logical consequence of the premises.

12. We use groud terms 0, $s(0)$, $s(s(0))$, ..., to represent natural numbers 0, 1, 2, ..., where 0 is a constants and s is a unary function such that $s(x) = x+1$ for all natural numbers x.

a) Let predicates $J2(x), J3(x)$ and $J6(x)$ represent that a natural number x is divisible by two, three and six, respectively. Define these predicates with sentences in predicate logic using the definitions of $J2$ and $J3$ to define $J6$.

b) Use resolution to prove that if a natural number x is divisible by two and three, then natural number $x+6$ is divisible by six.

Solution to Problem 12

We start with the base cases, that is, 0 is divisible by two and three:

$$J2(0),$$
$$J3(0).$$

Furthermore, divisibility for larger numbers:

$$\forall x(J2(x) \to J2(s(s(x)))),$$
$$\forall x(J3(x) \to J3(s(s(s(x))))).$$

Finally, divisibility by six:

$$\forall x(J2(x) \land J3(x) \to J6(x)).$$

We transform the sentences into clausal form. For the definition of predicate $J2(x)$ we get:

$$\begin{array}{l}\forall x(J2(x) \to J2(s(s(x)))) \\ \forall x(\neg J2(x) \lor J2(s(s(x))) \\ \{\neg J2(x), J2(s(s(x)))\}.\end{array}$$

Similarly for the definition of predicate $J3(x)$ we obtain $\{\neg J3(x), J3(s(s(s(x))))\}$. The sentence defining predicate $J6(x)$ results in the following:

$$\begin{array}{l}\forall x(J2(x) \land J3(x) \to J6(x)) \\ \forall x(\neg(J2(x) \land J3(x)) \lor J6(x)) \\ \forall x(\neg J2(x) \lor \neg J3(x) \lor J6(x)) \\ \{\neg J2(x), \neg J3(x), J6(x)\}.\end{array}$$

From the negation of the query we obtain the following three clauses:

$$\begin{array}{l}\neg\forall x(J2(x) \land J3(x) \to J6(s^6(x))) \\ \neg\forall x(\neg(J2(x) \land J3(x)) \lor J6(s^6(x))) \\ \neg\forall x(\neg J2(x) \lor \neg J3(x)) \lor J6(s^6(x))) \\ \exists x\neg(\neg J2(x) \lor \neg J3(x) \lor J6(s^6(x))) \\ \exists x(J2(x) \land J3(x) \land \neg J6(s^6(x))) \\ \{J2(c)\}, \{J3(c)\} \text{ and } \{\neg J6(s^6(c))\}.\end{array}$$

The resolution refutation:

1. $\{J2(c)\}$, P
2. $\{\neg J2(x_1), J2(s(s(x_1)))\}$, P
3. $\{J2(s(s(c)))\}$, 1 & 2, x_1/c
4. $\{\neg J2(x_2), J2(s(s(x_2)))\}$, P
5. $\{J2(s^4(c))\}$, 3 & 4, $x_2/s(s(c))$
6. $\{\neg J2(x_3), J2(s(s(x_3)))\}$, P
7. $\{J2(s^6(c))\}$, 5 & 6, $x_3/s^6(c)$
8. $\{J3(c)\}$, P
9. $\{\neg J3(x_4), J3(s(s(s(x_4))))\}$, P
10. $\{J3(s(s(s(c))))\}$, 8 & 9, x_4/c
11. $\{\neg J3(x_5), J3(s(s(s(x_5))))\}$, P
12. $\{J3(s^6(c))\}$, 10 & 11, $x_4/s(s(s(c)))$
13. $\{\neg J2(x_6), \neg J3(x_6), J6(x_6)\}$, P
14. $\{\neg J3(s^6(c)), J6(s^6(c))\}$, 7 & 13, $x_6/s^6(c)$
15. $\{J6(s^6(c))\}$, 12 & 14
16. $\{\neg J6(s^6(c))\}$, P
17. $\square$, 15 & 16