T-79.3001 Logic in Computer Science: Foundations **Spring 2009**
Exercise 8 ([Nerode and Shore, 1997], Predicate Logic, Chapters 4 and 9)
March 26 – March 30, 2009

Tutorial problems

1. Let $\mathcal{S}$ be a structure with a universe $U = \{0,1,2\}$ and the respective interpretations of function symbol f and a predicate symbol E as follows:

$$f^{\mathcal{S}}(n) = n+2 \text{ (mod 3)},$$
$$E^{\mathcal{S}} = \{\langle 0,1\rangle, \langle 1,0\rangle, \langle 1,2\rangle, \langle 2,2\rangle\}.$$

a) Draw the structure as a graph, where vertices are elements of U and E describes the edges.

b) Show using the truth definition, that $\mathcal{S} \models \forall x \exists y E(x,y)$.

c) Show using the truth definition, that $\mathcal{S} \models \exists x \neg E(x, f(x))$.

2. Show that the following statements are true:

a) $\{\forall x\big(A(x) \lor B(x)\big)\} \not\models \forall x A(x) \lor \forall x B(x)$.

b) $\forall x \exists y G(x,y) \not\equiv \exists y \forall x G(x,y)$.

3. Transform the following sentences into clausal form:

a) $\forall x \exists y \Big(P(x,y) \to \neg\big(Q(x,y) \lor \forall z Q(z,x)\big)\Big)$.

b) $\neg \exists x \Big(\exists y \big(A(x,y) \land B(y,x)\big) \to \neg \forall y B(y,x)\Big)$.

Demonstration problems

4. Let R be a binary predicate with interpretation $R^{\mathcal{S}} \subseteq U \times U$ (the set U is the domain of structure $\mathcal{S}$). In the following table we give definitions for some properties of relation $R^{\mathcal{S}}$.

Property	Definition
reflexivity	$\forall x R(x,x)$
irreflexivity	$\forall x \neg R(x,x)$
symmetry	$\forall x \forall y (R(x,y) \to R(y,x))$
asymmetry	$\forall x \forall y (R(x,y) \to \neg R(y,x))$
transitivity	$\forall x \forall y \forall z (R(x,y) \land R(y,z) \to R(x,z))$
seriality	$\forall x \exists y R(x,y)$

Consider a domain U consisting of people. Give examples of relations $R^{\mathcal{S}}$, ($\emptyset \subset R^{\mathcal{S}} \subset U^2$), that have properties described above.

5. Show that the following sentences are not valid by constructing a structure in which the sentence is false, i.e., construct a counter-example.

a) $\forall x \exists y P(x,y) \rightarrow \exists y \forall x P(x,y)$

b) $\exists x(P(x) \lor Q(x)) \rightarrow \exists x P(x) \land \exists x Q(x)$

c) $\neg \forall x(P(x) \rightarrow R(x)) \lor \neg \forall x(P(x) \rightarrow \neg R(x))$

6. Transform the following sentences into conjunctive normal form and perform skolemization.

a) $\forall y(\exists x P(x,y) \rightarrow \forall z Q(y,z)) \land \exists y(\forall x R(x,y) \lor \forall x Q(x,y))$

b) $\exists x \forall y R(x,y) \leftrightarrow \forall y \exists x P(x,y)$

c) $\forall x \exists y Q(x,y) \lor (\exists x \forall y P(x,y) \land \neg \exists x \exists y P(x,y))$

d) $\neg(\forall x \exists y P(x,y) \rightarrow \exists x \exists y R(x,y)) \land \forall x \neg \exists y Q(x,y)$

7. Use the rules in Lemma 9.1 [NS, 1997, page 129] to obtain rules for the following cases.

a) $\forall x \phi(x) \rightarrow \psi$

b) $\exists x \phi(x) \rightarrow \psi$

c) $\phi \rightarrow \forall x \psi(x)$

d) $\phi \rightarrow \exists x \psi(x)$

8. Transform the following sentences into clausal form.

a) $\neg \exists x((P(x) \rightarrow P(a)) \land (P(x) \rightarrow P(b)))$

b) $\forall y \exists x P(x,y)$

c) $\neg \forall y \exists x G(x,y)$

d) $\exists x \forall y \exists z(P(x,z) \lor P(z,y) \rightarrow G(x,y))$