T-79.3001 Logic in Computer Science: Foundations **Spring 2009**
Exercise 10 ([Nerode and Shore, 1997], Predicate Logic, Chapters 6 – 7)
April 16 – 20, 2009

Tutorial problems

1. Let strings consisting of letters a and b

"", "a", "b", "aa", "ab", "ba", "bb", ...

be represented by ground terms

e, $a(e)$, $b(e)$, $a(a(e))$, $a(b(e))$, $b(a(e))$, $b(b(e))$, ...

consisting of constant symbol e corresponding to the empty string "", and unary functions $a(x)$ and $b(x)$, which prepend the corresponding letter a or b to the string x. For example, $a(b(e))$ is interpreted as $a(b(\text{``''})) = a(\text{``}b\text{''}) =$ "ab".

Give a set of propositions Σ which defines the predicates

a) $\mathrm{P}(x,y) =$ "string x is the prefix of the string y", and

b) $\mathrm{S}(x,y) =$ "string x is the suffix of the string y".

2. Give a model $\mathcal{S} \models \Sigma$ on the basis of which it holds that

$$\Sigma \not\models \mathrm{P}(a(e),e).$$

3. Prove the following claims using semantic tableaux and the definitions you gave in the first problem:

a) $\Sigma \models \mathrm{P}(a(e),a(b(e)))$.

b) $\Sigma \models \mathrm{S}(b(e),a(b(e)))$.

Demonstration problems

4. Use semantic tableaux to see whether the following claims holds.

a) $\{\forall x\exists y(P(x) \to Q(y)), \forall xP(x)\} \models \forall xQ(x)$.

b) $\{\forall x\forall y(\exists z(R(x,z) \land R(z,y)) \to R(x,y)), R(a,b), R(b,a)\} \models R(a,a)$.

c) $\models \forall x\exists yR(x,y) \to (\forall y(\neg S(y) \to \neg\exists xR(x,y)) \to \exists xS(x))$.

5. We know that

(i) All guilty persons are liars.
(ii) At least one of the accused is also a witness.
(iii) No witness lies.

Use semantic tableaux to prove that all accused are not guilty.

6. We know that:

a) If a brick is on another brick, then it is not on the table.
b) Every brick is either on the table or on another brick.
c) No brick is on a brick which is also on some other brick.

Use semantic tableaux to prove that if a brick is on another brick, the other brick is on the table.

7. Let the nodes of a directed graph be represented as constant symbols $\{a, b, \ldots\}$ and the edges with binary predicate

$$K(x,y) = \text{“there is an edge from node } x \text{ to node } y\text{”}.$$

a) Define the predicates

$$R_n(x,y) = \text{“there is a directed path from } x \text{ to } y \text{ consisting of } n \text{ edges”},$$

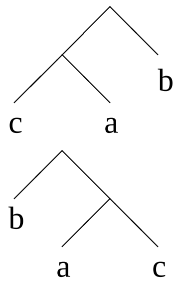
where $n \in \{1, 2, \ldots, k\}$. Describe the graph below using the predicate K.

$$a \; \overset{\longrightarrow}{\longleftarrow} \; b \longrightarrow c$$

b) Show using semantic tableaux that from the above description of the graph and the predicates R_2 and R_3 follows logically

$$\exists x (R_2(x,x) \wedge R_3(x,c)).$$

8. Let us represent binary trees using binary function symbol s (internal nodes) and unary function symbol l (leaf nodes). Hence the top tree of the below figure is represented by the term $s(s(l(c), l(a)), l(b))$.



a) Let $\mathrm{PK}(x,y)$ state that the binary tree x is the mirror image of the binary tree y. Define the predicate PK using predicate logic so that you can infer whether any two binary trees (represented as above) are mirror images of each other.

b) Show using semantic tableaux that the top binary tree is the mirror image of the bottom binary tree.

9. Quantifier $\exists!x$ means, that "there is exactly one x". The claim $\exists!x\phi x$ can be represented using predicate logic as follows:

$$(\exists x\phi(x)) \wedge (\forall x \forall y(\phi(x) \wedge \phi(y) \rightarrow x = y)).$$

Formalize the following sentences using predicate logic:

1. There is exactly one Father Christmas.
2. All Santa Clauses are Father Christmases.
3. All Father Christmases are Santa Claues.

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4. There is exactly one Santa Claus.

Show using semantic tableaux that the sentence number four is a logical consequences of the sentences one to three.