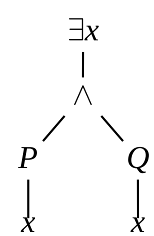
T-79.3001 Logic in computer science: foundations **Spring 2008**
Exercise 7 ([NS, 1997], Predicate Logic, Chapters 1–4)
March 19, and 27–28, 2008

Solutions to demonstration problems

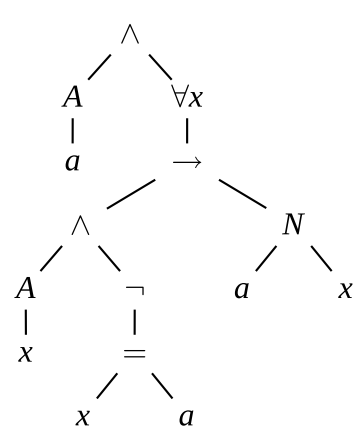
Solution to Problem 4

All of the solutions use either universal or existential quantifiers or both. If we want to say that some property $\phi(x)$ holds for all those x that have also property $P(x)$, we formalize that with: $\forall x(P(x) \to \phi(x))$. If some property $\phi(x)$ holds for some x that also satisfies $P(x)$, we use $\exists x(P(x) \wedge \phi(x))$. In the solutios many predicates (e.g. $P(x)$ in the first case) are used to denote the type of the x.

a) $\exists x(P(x) \wedge V(x))$, when
$P(x)$ = x is a gate.
$V(x)$ = x is faulty.



b) $A(a) \wedge (\forall x(A(x) \wedge \neg(x = a) \to N(a,x))$, when
a = the algorithm in question
$A(x)$ = x is an algorithm
$N(x,y)$ = x is faster than y.



c) $\forall x(K(x) \rightarrow \exists y(T(y) \wedge R(x,y)))$, when
$K(x)$ = x is a participant of the course.
$T(x)$ = x is a workstation
$R(x,y)$ = x uses y.

d) $\forall x(T(x) \rightarrow \forall y \forall z(P(y) \wedge P(z) \wedge K(y,x) \wedge K(z,x) \rightarrow y = z))$, when
$P(x)$ = x is a process.
$T(x)$ = x is a file.
$K(x,y)$ = x writes in y.

The above solutions are not the only possible ones.

Solution to Problem 5

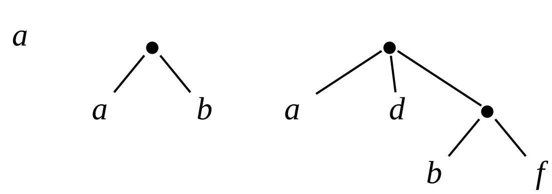
a) $\forall y(\exists x(P(x) \wedge Q(x)) \rightarrow L(x))$.

b) $\exists x \exists y(P(x,y) \vee Q(y,x)) \leftrightarrow \forall x \neg K(f(x))$

c) $\forall x \forall y(A \wedge B)$

Solution to Problem 6
Using the constant c and function f we get the set of terms $\{c, f(c), f^2(c), f^3(c), \dots\}$. More terms can be obtained using function g, arguments of g can be any pair from the previous set, for example $g(c,c)$ ja $g(f^3(c), f^{108}(c))$. Naturally these new terms can again be used as arguments for f and g, and we get, e.g, $f(g(f^5(c), f^{13}(c)))$ ja $g(g(c, f(c)), f^8(c))$. This process can be continued for arbitrarily many steps.

Solution to Problem 7
We represent trees as lists. Let constant e denote an empty list, and consider binary function $c \in \mathcal{F}_2$ (ensimmäinen argumentti listan ensimmäinen alkio ja toinen argumentti loput listasta), and unary function $l \in \mathcal{F}_1$ (lehtisolmu). Function $c(x,y)$ denotes a list: x is the first element in the list and y is the rest of the list. Function $l(x)$ denotes that x is a leaf node.
Consider the following trees:



The first of these is represented as $l(a)$, the second as $c(l(a), c(l(b), e))$ and the third as $c(l(a), c(l(d), c(c(l(b), c(l(f), e)), e)))$.

Solution to Problem 8

A sentence is a formula with no free occurences of any variable. We know that $\forall x \phi(x)$ is a sentence. $\phi(t)$ means a formula in which each free occurence of x is replaced with t. Since t is ground, also $\phi(t)$ is a sentence.

Solution to Problem 9

The pairs in $\mathbb{N}^2$ can be placed in an array as follows:

$$\begin{array}{ccccc}
\langle 0,0\rangle & \langle 0,1\rangle & \langle 0,2\rangle & \langle 0,3\rangle & \cdots \\
\langle 1,0\rangle & \langle 1,1\rangle & \langle 1,2\rangle & \langle 1,3\rangle & \cdots \\
\langle 2,0\rangle & \langle 2,1\rangle & \langle 2,2\rangle & \langle 2,3\rangle & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

The idea is the same as when showing that there are equally many elements in $\mathbb{N}^2$ and in $\mathbb{N}$, i.e., a bijective mapping from $\mathbb{N}$ to pairs is defined as $f(0) = \langle 0,0\rangle$ and working along diagonals for larger values, for instance, $f(1) = \langle 0,1\rangle$, $f(2) = \langle 1,0\rangle$ etc.

Now, we choose the interpretations as follows: $c^{\mathcal{S}} = \langle 0,0\rangle$, and

$$\begin{array}{ccccccc}
f(c)^{\mathcal{S}} & = & \langle 0,1\rangle & f(f(c))^{\mathcal{S}} & = & \langle 1,0\rangle \\
f^3(c)^{\mathcal{S}} & = & \langle 0,2\rangle & f^4(c)^{\mathcal{S}} & = & \langle 1,1\rangle \\
\vdots & & & \vdots & &
\end{array}$$

Thus $f^{\mathcal{S}}$ is

$$\begin{aligned}
f^{\mathcal{S}} : \langle x,y\rangle &\rightarrow \langle x',y'\rangle \\
x' &= g(x)(y+1) + (1-g(x))(x-1) \\
y' &= (1-g(x))(y+1)
\end{aligned}$$

where $g(x)$ is

$$g(x) = \begin{cases} 1, & \text{if } x = 0. \\ 0, & \text{otherwise.} \end{cases}$$

Solution to Problem 10

a) In the graphs we are particularly interested in edges, which we will denote by predicate $K(x,y)$ (there is a edge from node x to node y in the graph). There are several possible ways to denote the colors.

(i) We can fix the set of the colors and represent them as predicates. If there are n different colors in set C, we define predicates $C_1(x),\dots,C_n(n)$. A predicate $C_i(x)$ means that the node x is of the color C_i. The problem description demands that each node has a unique color and that if there is a edge between two nodes the nodes have different colors.

The first condition can be stated with a set of statements of the form:

$$\forall x(C_i(x) \leftrightarrow \neg C_1(x) \wedge \cdots \wedge \neg C_{i-1}(x) \wedge \neg C_{i+1} \wedge \cdots \wedge \neg C_n(x))$$

where $i = 1,\dots,n$ (notice that $\neg C_i(x)$ is not in the conjunction of the right side).

The second condition is formalized for each $C_i(x)$ as follows:

$$\forall x \forall y(K(x,y) \rightarrow (C_i(x) \rightarrow \neg C_i(y))).$$

(ii) The second possibility is to leave the definition of the colors open and use a predicate $V(x,y)$ (the node x is of the color y).

Now the uniqueness of node colors can be expressed as:

$$\forall x \forall y \forall z(V(x,y) \wedge V(x,z) \rightarrow y = z).$$

Informally, if a node x has both colors y and x, then the colors y and z must, in fact, be the same color.

The second condition can be expressed with:

$$\forall x \forall y \forall z(K(x,y) \rightarrow (V(x,z) \rightarrow \neg V(y,z)).$$

(iii) The third possibility is to define a function symbol v. Now $v(x)$ means the color of the node x. Because the value of a function is by definition unique, only the second condition has to be formalized:

$$\forall x \forall y(K(x,y) \rightarrow \neg(v(x) = v(y))).$$

b) Let's construct a model for the case (i) when $n = 2$. We will define a structure $\mathcal{S}$, where the universume is $U = \{a_1, a_2\}$ (two nodes). The interpretation of predicate K is $K^{\mathcal{S}} = \{\langle a_1, a_2\rangle, \langle a_2, a_1\rangle\}$ (there is a edge from node a_1 to a_2 and from a_2 to a_1).

The interpretation of the colors C_1 and C_2 are $C_1^{\mathcal{S}} = \{a_1\}$ and $C_2^{\mathcal{S}} = \{a_2\}$.

We now check that sentences

$$\forall x(C_1(x) \leftrightarrow \neg C_2(x))$$

$$\forall x \forall y(K(x,y) \to (C_1(x) \to \neg C_1(y))$$

and

$$\forall x \forall y(K(x,y) \to (C_2(x) \to \neg C_2(y))$$

are true in the structure. $\mathcal{S}$ (that is, $\mathcal{S}$ is a model for the sentences). The first of the sentences is equivalent to

$$\forall x(C_2(x) \leftrightarrow \neg C_1(x)),$$

which also belongs to the set of sentences when $n = 2$.

Now

$$\mathcal{S} \models \forall x(C_1(x) \leftrightarrow \neg C_2(x))$$

if and only if

$$\mathcal{S}[x \mapsto a_1] \models (C_1(x) \leftrightarrow \neg C_2(x)) \qquad \text{and} \qquad \mathcal{S}[x \mapsto a_2] \models (C_1(x) \leftrightarrow \neg C_2(x))$$

Since $a_1 \in C_1^{\mathcal{S}}$, we have $\mathcal{S}[x \mapsto a_1] \models C_1(x)$. Also, since $a_1 \notin C_2^{\mathcal{S}}$, it holds $\mathcal{S}[x \mapsto a_1] \not\models C_2(x)$. Thus

$$\mathcal{S}[x \mapsto a_1] \models (C_1(x) \leftrightarrow \neg C_2(x))$$

Similarly we show $\mathcal{S}[x \mapsto a_2] \models (C_1(x) \leftrightarrow \neg C_2(x))$, and $\mathcal{S} \models \forall x(C_1(x) \leftrightarrow \neg C_2(x))$ follows.

Now $\mathcal{S} \models \forall x \forall y(K(x,y) \to (C_1(x) \to \neg C_1(y))$ if and only if

$$K(x,y) \to (C_1(x) \to \neg C_1(y))$$

is true in

$$\begin{array}{lcl} \mathcal{S}[x \mapsto a_1, y \mapsto a_1], & & \mathcal{S}[x \mapsto a_1, y \mapsto a_2], \\ \mathcal{S}[x \mapsto a_2, y \mapsto a_1] & \text{ja} & \mathcal{S}[x \mapsto a_2, y \mapsto a_2]. \end{array}$$

Because pairs $\langle a_1, a_1 \rangle$ and $\langle a_2, a_2 \rangle$ don't belong to $K^{\mathcal{S}}$, atomic sentence $K(x,y)$ is false in the first and the last case, and then $K(x,y) \to (C_1(x) \to \neg C_1(y))$ is true in these cases. Since pair $\langle a_1, a_2 \rangle$ belongs to $K^{\mathcal{S}}$, $\mathcal{S}[x \mapsto a_1, y \mapsto a_2] \models K(x,y)$ and the proposition is true for $\mathcal{S}[x \mapsto a_1, y \mapsto a_2]$ if and only if $\mathcal{S}[x \mapsto a_1, y \mapsto a_2] \models C_1(x) \to \neg C_1(y)$. This holds, because $a_1 \in C_1^{\mathcal{S}}$ and $a_2 \notin C_2^{\mathcal{S}}$, and therefore $\mathcal{S}[x \mapsto a_1, y \mapsto a_2] \models C_1(x)$ and $\mathcal{S}[x \mapsto a_1, y \mapsto a_2] \models \neg C_2(y)$. The proposition is also true in the third case. Difference to the previous one is that implication $C_1(x) \to \neg C_1(y)$ is true in $\mathcal{S}$, because $\mathcal{S}[x \mapsto a_2, y \mapsto a_1] \not\models C_1(x)$. Thus $\mathcal{S}$ is a model for $\forall x \forall y(K(x,y) \to (C_1(x) \to \neg C_1(y))$.

Because the sentences are symmetrical, $\mathcal{S}$ is also a model for sentence

$$\forall x \forall y (K(x,y) \rightarrow (C_2(x) \rightarrow \neg C_2(y))).$$

The models will be more complex, if the colors are implemented according to formalizations (ii) or (iii).

c) We will define a structure $\mathcal{S}$ when $n = 2$, where the set of sentences is not satisfiable. We will choose as the universume $U = \{a\}$ (there is only one node) and the interpretation of predicate $K^{\mathcal{S}} = \{\langle a, a \rangle\}$. Now

$$\forall x (C_1(x) \leftrightarrow \neg C_2(x))$$

is not satisfied in structure $\mathcal{S}$, if

$$\mathcal{S}[x \mapsto a] \not\models C_1(x) \leftrightarrow \neg C_2(x).$$

So we can construct the interpretations of the color predicates as follows:

$$\mathcal{S}[x \mapsto a] \models C_1(x) \qquad \text{and} \qquad \mathcal{S}[x \mapsto a] \models C_2(x)$$

choosing

$$C_1^{\mathcal{S}} = C_2^{\mathcal{S}} = \{a\}$$

Now $\mathcal{S}$ cannot be a model for the set of sentences.