T-79.3001 Logic in computer science: foundations **Spring 2008**
Exercise 6 ([Nerode and Shore, 1997], Chapter 8)
March 5, and 13–14, 2008

Solutions to demonstration problems

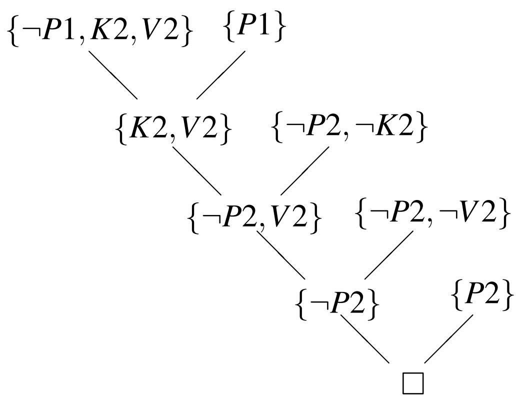
Solution to Problem 4

We transform the propositions into CNF and clauses. The last proposition in the table is the negation of statement "both red lights are not on at the same time", that is,

$$\neg(\neg(P1 \land P2)) \equiv P1 \land P2.$$

$Pi \lor Ki \lor Vi$		$\{Pi, Ki, Vi\}$
$Pi \to \neg Ki \land \neg Vi$	$\equiv \neg Pi \lor (\neg Ki \land \neg Vi)$ $\equiv (\neg Pi \lor \neg Ki) \land (\neg Pi \lor \neg Vi)$	$\{\neg Pi, \neg Ki\}, \{\neg Pi, \neg Vi\}$
$Ki \to \neg Pi \land \neg Vi$	$\equiv (\neg Ki \lor \neg Pi) \land (\neg Ki \lor \neg Vi)$	$\{\neg Pi, \neg Ki\}, \{\neg Ki, \neg Vi\}$
$Vi \to \neg Pi \land \neg Ki$	$\equiv (\neg Vi \lor \neg Pi) \land (\neg Vi \lor \neg Ki)$	$\{\neg Pi, \neg Vi\}, \{\neg Ki, \neg Vi\}$
$\neg(V1 \land V2)$	$\equiv \neg V1 \lor \neg V2$	$\{\neg V1, \neg V2\}$
$P1 \to (K2 \lor V2)$	$\equiv \neg P1 \lor K2 \lor V2$	$\{\neg P1, K2, V2\}$
$P2 \to (K1 \lor V1)$	$\equiv \neg P2 \lor K1 \lor V1$	$\{\neg P2, K1, V1\}$
$P1 \land P2$		$\{P1\}, \{P2\}$

We show that the set of clauses given in the table is unsatisfiable (empty clause $\square$ means contradiction), which implies that $\neg(P1 \land P2)$ is derivable from the other clauses.



Solution to Problem 5

The chemical reactions can be formalized as implications, which can then be transformed into clausul form. The resulting clauses are:

(1)

$$
\begin{aligned}
& MgO + H_2 \rightarrow Mg + H_2O \\
\Longrightarrow & MgO \wedge H_2 \rightarrow Mg \wedge H_2O \\
\Longrightarrow & \neg MgO \vee \neg H_2 \vee (Mg \wedge H_2O) \\
\Longrightarrow & (\neg MgO \vee \neg H_2 \vee Mg) \wedge (\neg MgO \vee \neg H_2 \vee H_2O)
\end{aligned}
$$

The first reaction results in two clauses: : $\{\neg MgO, \neg H_2, Mg\}$ and $\{\neg MgO, \neg H_2, H_2O\}$.

(2)

$$
\begin{aligned}
& C + O_2 \rightarrow CO_2 \\
\Longrightarrow & C \wedge O_2 \rightarrow CO_2 \\
\Longrightarrow & \neg C \vee \neg O_2 \vee CO_2 \\
\Longrightarrow & \{\neg C, \neg O_2, CO_2\}
\end{aligned}
$$

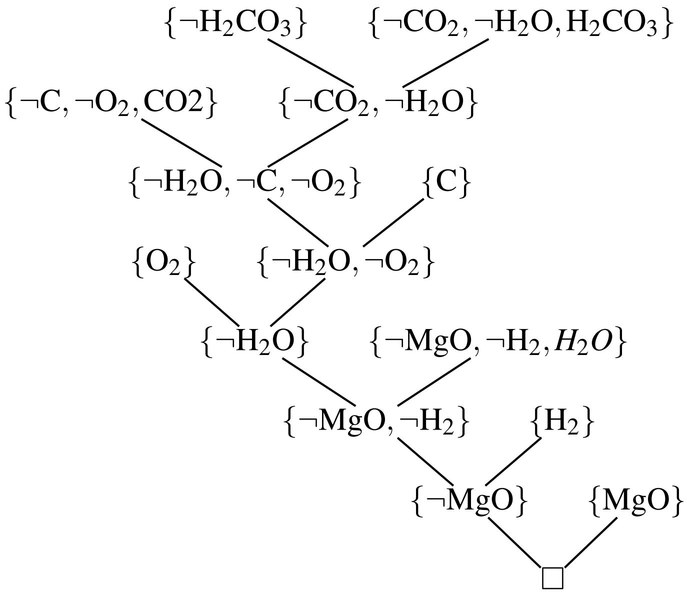
(3)

$$
\begin{aligned}
& CO_2 + H_2O \rightarrow H_2CO_3 \\
\Longrightarrow & CO_2 \wedge H_2O \rightarrow H_2CO_3 \\
\Longrightarrow & \neg CO_2 \vee \neg H_2O \vee H_2CO_3 \\
\Longrightarrow & \{\neg CO_2, \neg H_2O, H_2CO_3\}
\end{aligned}
$$

The elements availabe at the start are:

$$
\begin{aligned}
& MgO \wedge H_2 \wedge O_2 \wedge C \\
\Longrightarrow & \{MgO\}, \{H_2\}, \{O_2\}\{C\}
\end{aligned}
$$

We denote the above set of clauses with Σ. now we want to prove that $\Sigma \models H_2CO_3$. The proof is constructed by showing that $\Sigma \cup \{\neg H_2CO_3\}$ is unsatisfiable.



Solution to Problem 6

The solution is obtained from "Computational Complexity" by C. Papadimitriou. A deterministic Turing machine is a quadruple $\langle A, S, s_0, t\rangle$, where

- A is the alphabet,
- S is the set of states,
- $t : S \times A \to S \times A \times \{\to, \leftarrow, \downarrow\}$ is the state transition function
- $s_0 \in S$ is the start state.

For our machine we have $S = \{s\}$, $A = \{0, 1\}$, $s_0 = s$ and the state transition function is given in the following table:

$p \in S$	$\sigma \in A$	$t(p, \sigma)$
s	0	$(h, 1, -)$
s	1	$(s, 0, \to)$
s	$\sqcup$	$(h, 1, -)$
s	$\triangleright$	$(s, \triangleright, \to)$

With input 1101 the computation goes as follows: $(s, \triangleright, 1101) \xrightarrow{M} (s, \triangleright 0, 101)$ $\xrightarrow{M} (s, \triangleright 00, 01) \xrightarrow{M} (h, \triangleright 001, 1)$.

Solution to Problem 7

The problem of 3-coloring a graph is as follows: "give a graph G, is there a way to color the nodes in G using 3 colors so that no two adjacent nodes have same color?"

Let $N = \{n_1, n_2, \ldots, n_m\}$ be the set of nodes and $E \subseteq N \times N$ the set of edges.

For each node n_i we take atomic propositions $R_{n_i}, G_{n_i}, B_{n_i}$ to denote that node n_i is colored red, green or blue, respectively.

Each node is colored with some color, that is, $R_{n_i} \vee G_{n_i} \vee B_{n_i}$, for each n_i.

No node is colored with two different colors, that is,

$$(R_{n_i} \rightarrow (\neg G_{n_i} \wedge \neg B_{n_i})) \wedge (G_{n_i} \rightarrow (\neg R_{n_i} \wedge \neg B_{n_i})) \wedge (B_{n_i} \rightarrow (\neg R_{n_i} \wedge \neg G_{n_i})),$$

for each n_i.

Finally, two adjacent color can't have same color, that is,

$$(R_n \rightarrow \neg R_m) \wedge (G_n \rightarrow \neg G_m) \wedge (B_n \rightarrow \neg B_m),$$

for each $(n, m) \in E$.

Now, if we take the conjunction of all these propositions (denoted by ϕ), then ϕ is satisfiable iff the graph has a 3-coloring (the proof is omitted).