T-79.3001 Logic in computer science: foundations **Spring 2008**
Exercise 4 ([NS 1997], Chapter I, Sections 4 and 7)
February 20–22, 2008

Solutions to demonstration problems

Solution to Problem 4

Based on the definition and the semantic tableaux rules for basic connectives, we get the following rules for Peirce arrow:

$$\begin{array}{c} T(A \downarrow B) \\ | \\ F(A) \\ | \\ F(B) \end{array} \qquad\qquad \begin{array}{ccc} & F(A \downarrow B) & \\ / & & \backslash \\ T(A) & & T(B) \end{array}$$

Solution to Problem 5

We will proceed by constructing semantic tableaux for the negations of the propositions ($E(\phi)$). If all branches close (that is, there are contradictions) then ϕ is valid. If a branch is closed before the tableau is ready, then it is not necessary to continue working on that branch.

You should notice, that the semantic tableu is actually used to find models for $\neg\phi$. If all branches are contradictionary, then $\neg\phi$ doesn't have a model and its negation is valid.

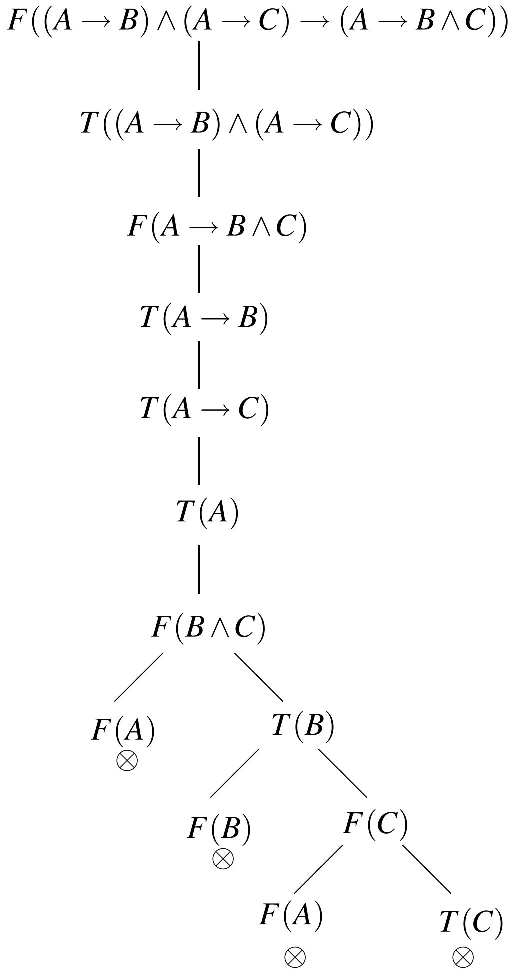
a) $A \to (B \to B)$:

$$\begin{array}{c} F(A \to (B \to B)) \\ | \\ T(A) \\ | \\ F(B \to B) \\ | \\ T(B) \\ | \\ F(B) \\ \otimes \end{array}$$

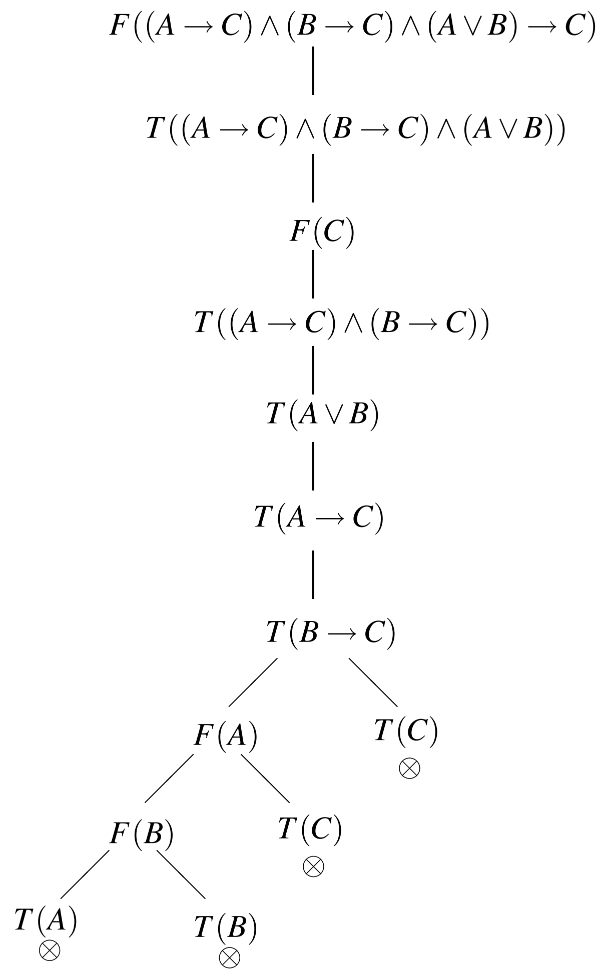
b) $(A \to B) \land (B \to C) \to (A \to C)$:

$$\begin{tikzcd}[column sep=small]
 & F((A \to B) \land (B \to C) \to (A \to C)) \arrow[d, dash] & \\
 & T((A \to B) \land (B \to C)) \arrow[d, dash] & \\
 & F(A \to C) \arrow[d, dash] & \\
 & T(A \to B) \arrow[d, dash] & \\
 & T(B \to C) \arrow[d, dash] & \\
 & T(A) \arrow[d, dash] & \\
 & F(C) \arrow[dl, dash] \arrow[dr, dash] & \\
F(A) \; \otimes & & T(B) \arrow[dl, dash] \arrow[dr, dash] & \\
 & F(B) \; \otimes & & T(C) \; \otimes
\end{tikzcd}$$

c) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$:



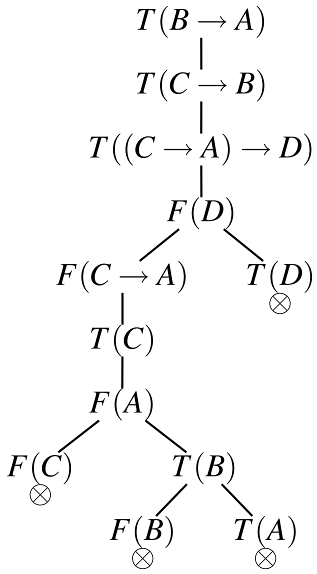
d) $(A \rightarrow C) \wedge (B \rightarrow C) \wedge (A \vee B) \rightarrow C$:



Solution to Problem 6

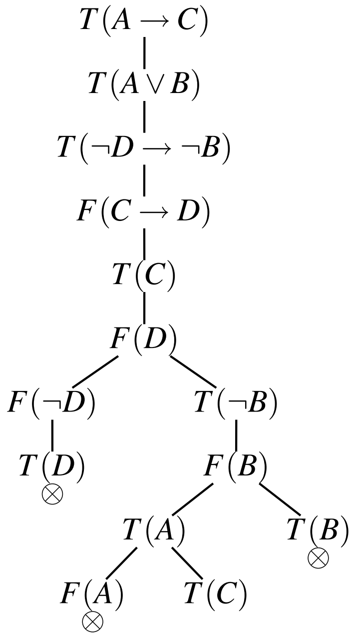
When we are checking whether a proposition P is a logical consequence of a set of propositions S we put all node $T(\alpha)$ to the semantic tableaux for all $\alpha \in S$. Next we add $F(P)$ to the tableaux and use inference rules to complete it. If all branches of the tableaux end in a contradiction, we know that P can't be false if all propositions in S are true and so P is a logical consequence. Otherwise, the claim doesn't hold and we can construct a counterexample from an uncontradictionary branch.

a)



As all brances are contradictory, D is a logical consequence of the set.

b)



As there is an unclosed branch, $C \to D$ is not logical consequence of the set. We can construct a counter example from the open branch: $\mathcal{A} = \{A, C\}$. Thus it holds $\mathcal{A} \models A \to C$, $\mathcal{A} \models A \vee B$, $\mathcal{A} \models \neg D \to \neg B$, ja $\mathcal{A} \not\models C \to D$ (check!).

c) $\models \phi$ denotes that ϕ is valid. To prove this we construct a semantic tableuax for $F(\phi)$.

$$F((A \to (B \to C)) \to ((A \to C) \to (A \to B)))$$
$$T(A \to (B \to C))$$
$$F((A \to C) \to (A \to B))$$
$$T(A \to C)$$
$$F(A \to B)$$
$$T(A)$$
$$F(B)$$
$$\begin{array}{ccc} F(A) & & T(C) \\ \otimes & F(A) & T(B \to C)^{[6]} \\ & \otimes & F(B) \quad T(C) \end{array}$$

Since there is an unclosed brach, the proposition is not valid. A counter example can be constructed from an open branch, for example from the rightmost open branch we get: $\mathcal{A} = \{A, C\}$.

d) $$F((\neg B \to (A \to C)) \to (A \to B \vee C))$$
$$T(\neg B \to (A \to C))$$
$$F(A \to B \vee C)$$
$$T(A)$$
$$F(B \vee C)$$
$$F(B)$$
$$F(C)$$
$$\begin{array}{ccc} F(\neg B) & & T(A \to C) \\ T(B) & F(A) & T(C) \\ \otimes & \otimes & \otimes \end{array}$$

As all brances are contradictory, the proposition is valid.

Solution to Problem 7

$T(P1 \vee K1 \vee V1)$

$T(P1 \rightarrow \neg K1 \wedge \neg V1)$

$T(K1 \rightarrow \neg P1 \wedge \neg V1)$

$T(V1 \rightarrow \neg P1 \wedge \neg K1)$

$T(P2 \vee K2 \vee V2)$

$T(P2 \rightarrow \neg K2 \wedge \neg V2)$

$T(K2 \rightarrow \neg P2 \wedge \neg V2)$

$T(V2 \rightarrow \neg P2 \wedge \neg K2)$

$T(\neg(V1 \wedge V2))$

$T(P1 \rightarrow (K2 \vee V2))$

$T(P2 \rightarrow (K1 \vee V1))$

$F(\neg(P1 \wedge P2))$

$T(P1 \wedge P2)$

$T(P1)$

$T(P2)$

- $F(P1)$ $\otimes$
- $T(K2 \vee V2)$
  - $T(K2)$
    - $F(K2)$ $\otimes$
    - $T(\neg P2 \wedge \neg V2)$, $T(\neg P2)$, $T(\neg V2)$, $F(P2)$ $\otimes$
  - $T(V2)$
    - $F(K2)$
      - $F(V2)$ $\otimes$
      - $T(\neg P2 \wedge \neg K2)$, $T(\neg P2)$, $T(\neg K2)$, $F(P2)$ $\otimes$
    - $T(\neg P2 \wedge \neg V2)$, $T(\neg P2)$, $T(\neg V2)$, $F(P2)$ $\otimes$

Solution to Problem 8

a)

1.	$(P \to ((P \to P) \to P))$	[A1] $\alpha = P,\ \beta = P \to P$
2.	$((P \to ((P \to P) \to P)) \to$ $((P \to (P \to P)) \to (P \to P)))$	[A2] $\alpha = \gamma = P,\ \beta = P \to P$
3.	$((P \to (P \to P)) \to (P \to P))$	[MP:1,2]
4.	$(P \to (P \to P))$	[A1] $\alpha = P,\ \beta = P$
5.	$(P \to P)$	[MP:3,4]

b)

1.	$(Q \to R)$	[P2]
2.	$((Q \to R) \to (P \to (Q \to R)))$	[A1] $\alpha = Q \to R,\ \beta = P$
3.	$(P \to (Q \to R))$	[MP:1,2]
4.	$((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$	[A2] $\alpha = P,\ \beta = Q,\ \gamma = R$
5.	$((P \to Q) \to (P \to R))$	[MP:3,4]
6.	$(P \to Q)$	[P1]
7.	$(P \to R)$	[MP:5,6]

c)

1.	P	[P1]
2.	$(Q \to (P \to R))$	[P2]
3.	$(P \to (Q \to P))$	[A1] $\alpha = P,\ \beta = Q$
4.	$(Q \to P)$	[MP:1,3]
5.	$((Q \to (P \to R)) \to ((Q \to P) \to (Q \to R)))$	[A2] $\alpha = Q,\ \beta = P,\ \gamma = R$
6.	$((Q \to P) \to (Q \to R))$	[MP:2,5]
7.	$(Q \to R)$	[MP:4,6]