Solutions to demonstration problems

Solution to Problem 4

a) Assume that for some \( \phi \) it holds \( \Sigma \models \phi \) and \( \Sigma \models \neg \phi \). We use proof by contradiction, that is, we assume that \( \Sigma \) is satisfiable and show that this leads to contradiction. If \( \Sigma \) is satisfiable then there is a truth assignment \( \mathcal{A} \) such that for all \( \sigma \in \Sigma \), \( \mathcal{A} \models \sigma \). Since \( \Sigma \models \phi \), it holds \( \mathcal{A} \models \phi \). On the other hand, since \( \Sigma \models \neg \phi \), it holds \( \mathcal{A} \models \neg \phi \), which is equivalent to \( \mathcal{A} \not\models \phi \). Since no proposition can be true and false at the same time, this is a contradiction and the original claim holds, that is, \( \Sigma \) is unsatisfiable. \( \Box \)

b) Let \( \mathcal{A} \) be the only model for \( \Sigma \). For each proposition \( \phi \) it holds that \( \phi \) is either true in \( \mathcal{A} \) or \( \phi \) is false in \( \mathcal{A} \), that is, either \( \mathcal{A} \models \phi \) or \( \mathcal{A} \not\models \phi \) (equivalently \( \mathcal{A} \models \neg \phi \)). If \( \mathcal{A} \models \phi \), it holds \( \Sigma \models \phi \), and if \( \mathcal{A} \models \neg \phi \), it holds \( \Sigma \models \neg \phi \). \( \Box \)

Solution to Problem 5

\( C_n(\Sigma) \) denotes the set of logical consequences of a set of propositions \( \Sigma \), that is, \( C_n(\Sigma) = \{ \phi \mid \Sigma \models \phi \} \).

a) Assume that \( \Sigma \not\subseteq C_n(\Sigma) \). Then \( \Sigma \) contains \( \alpha \) such that there is a model \( \mathcal{A} \) of \( \Sigma \) that is not a model of \( \alpha \), that is, \( \mathcal{A} \not\models \alpha \). On the other hand, since \( \mathcal{A} \) is a model of \( \Sigma \) it holds \( \mathcal{A} \models \sigma \) for all \( \sigma \in \Sigma \). Since \( \alpha \in \Sigma \), we have \( \mathcal{A} \models \alpha \). This is a contradiction, and thus \( \Sigma \subseteq C_n(\Sigma) \). \( \Box \)

b) Consider arbitrary \( \alpha \in C_n(\Sigma_1) \). \( \alpha \) is true in all the models of \( \Sigma_1 \), that is, in all the truth assignments in which all the propositions in \( \Sigma_1 \) are true. Because \( \Sigma_1 \subseteq \Sigma_2 \), every model of \( \Sigma_2 \) is also a model of \( \Sigma_1 \). This implies that \( \alpha \) is true in every model of \( \Sigma_2 \), that is, \( \alpha \in C_n(\Sigma_2) \). \( \Box \)

c) Assume that \( \Sigma \models \phi \), that is, for all \( \mathcal{A} \) such that \( \mathcal{A} \models \sigma \) for all \( \sigma \in \Sigma \), it holds \( \mathcal{A} \not\models \phi \). Based on item b) it holds \( C_n(\Sigma) \subseteq C_n(\Sigma \cup \{ \phi \}) \) and it suffices to show that \( C_n(\Sigma \cup \{ \phi \}) \subseteq C_n(\Sigma) \). Consider arbitrary \( \alpha \in C_n(\Sigma \cup \{ \phi \}) \). It holds \( \Sigma \cup \{ \phi \} \models \alpha \), that is, \( \alpha \) is true in every model of \( \Sigma \cup \{ \phi \} \). But these are exactly the same as the models of \( \Sigma \), that is, \( \Sigma \models \alpha \) and \( \alpha \in C_n(\Sigma) \). \( \Box \)
Solution to Problem 6
We choose the following atomical propositions.

\[
A = \text{“person 1 votes yes”} \\
B = \text{“person 2 votes yes”} \\
C = \text{“person 3 votes yes”} \\
Y = \text{“majority of yes-votes”}
\]

Two yes-votes results in majority for yes.

\[
A \land B \rightarrow Y \\
A \land C \rightarrow Y \\
B \land C \rightarrow Y
\]

Two no-votes results in minority of yes votes.

\[
\neg A \land \neg B \rightarrow \neg Y \\
\neg A \land \neg C \rightarrow \neg Y \\
\neg B \land \neg C \rightarrow \neg Y
\]

When there are three persons and a chairperson, we take in addition the following atomical propositions.

\[
P = \text{“chair votes yes”} \\
IC = \text{“result of the vote depends on the vote of the chair”}
\]

Three yes or no votes gives the result directly.

\[
A \land B \land C \rightarrow Y \\
\neg A \land \neg B \land \neg C \rightarrow \neg Y
\]

Otherwise, the vote of the chairperson impacts the outcome of the vote.

\[
A \land \neg B \land \neg C \rightarrow IC \\
\neg A \land B \land \neg C \rightarrow IC \\
\neg A \land \neg B \land C \rightarrow IC \\
A \land B \land \neg C \rightarrow IC \\
A \land \neg B \land C \rightarrow IC \\
\neg A \land B \land C \rightarrow IC
\]

The impact of the chairperson’s vote.

\[
IC \land P \rightarrow Y \\
IC \land \neg P \rightarrow \neg Y
\]

Naturally, there are also several other possibilities how to model the voting system.

Solution to Problem 7
Choose for example the following atomical propositions.

\[
A = \text{“a valid period ticket on the card”} \\
B = \text{“a valid value ticket on the card”} \\
C = \text{“a valid transfer ticket on the card”} \\
V = \text{“green light in the reader”} \\
K = \text{“yellow light in the reader”} \\
P = \text{“red light in the reader”}
\]

\[
D = \text{“period } \leq 3 \text{ days”} \\
E = \text{“value } \leq 5 \text{ euros”} \\
F = \text{“other error”}
\]
Now the system can be formalized for example as follows.

1. $A \lor B \lor C \rightarrow V$
2. $D \lor E \rightarrow K \land V$
3. $\neg A \lor \neg C \lor F \rightarrow P$