T-79.3001 Logic in computer science: foundations **Spring 2008**
Exercise 3 ([NS 1997], Chapter I, Section 3)
February 13–15, 2008

Solutions to demonstration problems

Solution to Problem 4

a) Assume that for some ϕ it holds $\Sigma \models \phi$ and $\Sigma \models \neg\phi$. We use **proof by contradiction**, that is, we assume that Σ is satisfiable and show that this leads to contradiction. If Σ is satisfiable then there is a truth assignment $\mathcal{A}$ such that for all $\sigma \in \Sigma$, $\mathcal{A} \models \sigma$. Since $\Sigma \models \phi$, it holds $\mathcal{A} \models \phi$. On the other hand, since $\Sigma \models \neg\phi$, it holds $\mathcal{A} \models \neg\phi$, which is equivalent to $\mathcal{A} \not\models \phi$. Since no proposition can be true and false at the same time, this is a contradiction and the original claim holds, that is, Σ is unsatisfiable. $\square$

b) Let $\mathcal{A}$ be the only model for Σ. For each proposition ϕ it holds that ϕ is **either** true in $\mathcal{A}$ **or** ϕ is false in $\mathcal{A}$, that is, either $\mathcal{A} \models \phi$ or $\mathcal{A} \not\models \phi$ (equivalently $\mathcal{A} \models \neg\phi$). If $\mathcal{A} \models \phi$, it holds $\Sigma \models \phi$, and if $\mathcal{A} \models \neg\phi$, it holds $\Sigma \models \neg\phi$. $\square$

Solution to Problem 5

$\mathrm{Cn}(\Sigma)$ denotes the set of logical consequences of a set of propositions Σ, that is, $\mathrm{Cn}(\Sigma) = \{\phi \mid \Sigma \models \phi\}$.

a) Assume that $\Sigma \not\subseteq \mathrm{Cn}(\Sigma)$. Then Σ contains α such that there is a model $\mathcal{A}$ of Σ that is not a model of α, that is, $\mathcal{A} \not\models \alpha$. On the other hand, since $\mathcal{A}$ is a model of Σ it holds $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$. Since $\alpha \in \Sigma$, we have $\mathcal{A} \models \alpha$. This is a contradiction, and thus $\Sigma \subseteq \mathrm{Cn}(\Sigma)$. $\square$

b) Consider arbitrary $\alpha \in \mathrm{Cn}(\Sigma_1)$. α is true in all the models of Σ_1, that is, in all the truth assignments in which all the propositions in Σ_1 are true. Because $\Sigma_1 \subseteq \Sigma_2$, every model of Σ_2 is also a model of Σ_1. This implies that α is true in every model of Σ_2, that is, $\alpha \in \mathrm{Cn}(\Sigma_2)$. $\square$

c) Assume that $\Sigma \models \phi$, that is, for all $\mathcal{A}$ such that $\mathcal{A} \models \sigma$ for all $\sigma \in \Sigma$, it holds $\mathcal{A} \models \phi$. Based on item b) it holds $\mathrm{Cn}(\Sigma) \subseteq \mathrm{Cn}(\Sigma \cup \{\phi\})$ and it suffices to show that $\mathrm{Cn}(\Sigma \cup \{\phi\}) \subseteq \mathrm{Cn}(\Sigma)$. Consider arbitrary $\alpha \in \mathrm{Cn}(\Sigma \cup \{\phi\})$. It holds $\Sigma \cup \{\phi\} \models \alpha$, that is, α is true in every model of $\Sigma \cup \{\phi\}$. But these are exactly the same as the models of Σ, that is, $\Sigma \models \alpha$ and $\alpha \in \mathrm{Cn}(\Sigma)$. $\square$

Solution to Problem 6
We choose the following atomical propositions.

$$\begin{aligned} A &= \text{“person 1 votes yes”} \\ B &= \text{“person 2 votes yes”} \\ C &= \text{“person 3 votes yes”} \\ Y &= \text{“majority of yes-votes”} \end{aligned}$$

Two yes-votes results in majority for yes.

$$A \wedge B \rightarrow Y \quad A \wedge C \rightarrow Y \quad B \wedge C \rightarrow Y$$

Two no-votes results in minority of yes votes.

$$\neg A \wedge \neg B \rightarrow \neg Y \quad \neg A \wedge \neg C \rightarrow \neg Y \quad \neg B \wedge \neg C \rightarrow \neg Y$$

When there are three persons and a chairperson, we take in addition the following atomical propositions.

$$\begin{aligned} P &= \text{“chair votes yes”} \\ IC &= \text{“result of the vote depends on the vote of the chair”} \end{aligned}$$

Three yes or no votes gives the result directly.

$$A \wedge B \wedge C \rightarrow Y \quad \neg A \wedge \neg B \wedge \neg C \rightarrow \neg Y$$

Otherwise, the vote of the chairperson impacts the outcome of the vote.

$$\begin{array}{ccc} A \wedge \neg B \wedge \neg C \rightarrow IC & \neg A \wedge B \wedge \neg C \rightarrow IC & \neg A \wedge \neg B \wedge C \rightarrow IC \\ A \wedge B \wedge \neg C \rightarrow IC & A \wedge \neg B \wedge C \rightarrow IC & \neg A \wedge B \wedge C \rightarrow IC \end{array}$$

The impact of the chairperson's vote.

$$IC \wedge P \rightarrow Y \quad IC \wedge \neg P \rightarrow \neg Y$$

Naturally, there are also several other possibilities how to model the voting system.

Solution to Problem 7
Choose for example the following atomical propositions.

$$\begin{array}{lcl lcl} A & = & \text{“a valid period ticket on the card”} & D & = & \text{“period} \leq 3 \text{ days”} \\ B & = & \text{“a valid value ticket on the card”} & E & = & \text{“value} \leq 5 \text{ euros”} \\ C & = & \text{“a valid transfer ticket on the card”} & F & = & \text{“other error”} \end{array}$$

$$\begin{aligned} V &= \text{“green light in the reader”} \\ K &= \text{“yellow light in the reader”} \\ P &= \text{“red light in the reader”} \end{aligned}$$

Now the system can be formalized for example as follows.

1. $A \vee B \vee C \rightarrow V$
2. $D \vee E \rightarrow K \wedge V$
3. $\neg A \vee \neg C \vee F \rightarrow P$