Tutorial problems

1. Define predicate $Y(x,y)$ (there is a connection from city $x$ to city $y$) using the predicate $L(x,y)$ (there is a flight from city $x$ to city $y$).

2. Show that the following sentences are not valid by constructing a structure in which the sentence is false, i.e., construct a counter-example.
   a) $\forall x(P(x) \rightarrow R(x)) \land \forall x(Q(x) \rightarrow R(x)) \rightarrow \forall x(P(x) \rightarrow Q(x))$
   b) $\forall x \forall y(R(x,y) \rightarrow R(y,x)) \rightarrow \exists y \forall x R(x,y)$

3. Transform the following sentences into clausal form.
   a) $\neg (\exists x A(x) \lor \exists x B(x) \rightarrow \exists x (A(x) \lor B(x)))$
   b) $(\forall x P(x) \rightarrow \exists x \forall y Q(x,y)) \rightarrow \neg \forall y P(y))$

Demonstration problems

4. Let $R$ be a binary predicate with interpretation $R^S \subseteq U \times U$ (the set $U$ is the domain of structure $S$). In the following table we give definitions for some properties of relation $R^S$.

<table>
<thead>
<tr>
<th>Property</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>reflexivity</td>
<td>$\forall x R(x,x)$</td>
</tr>
<tr>
<td>irreflexivity</td>
<td>$\forall x \neg R(x,x)$</td>
</tr>
<tr>
<td>symmetry</td>
<td>$\forall x \forall y (R(x,y) \rightarrow R(y,x))$</td>
</tr>
<tr>
<td>asymmetry</td>
<td>$\forall x \forall y (R(x,y) \rightarrow \neg R(y,x))$</td>
</tr>
<tr>
<td>transitivity</td>
<td>$\forall x \forall y \forall z (R(x,y) \land R(y,z) \rightarrow R(x,z))$</td>
</tr>
<tr>
<td>seriality</td>
<td>$\forall x \exists y R(x,y)$</td>
</tr>
</tbody>
</table>

Consider a domain $U$ consisting of people. Give examples of relations $R^S$, ($\emptyset \subset R^S \subset U^2$), that have properties described above.

5. Show that the following sentences are not valid by constructing a structure in which the sentence is false, i.e., construct a counter-example.
   a) $\forall x \exists y P(x,y) \rightarrow \exists y \forall x P(x,y)$
6. Transform the following sentences into conjunctive normal form and perform skolemization.

a) \( \forall y (\exists x P(x) \rightarrow \forall z Q(y, z)) \land \exists y (\forall x R(x, y) \lor \forall x Q(x, y)) \)

b) \( \exists x \forall y R(x, y) \leftrightarrow \forall y \exists x P(x, y) \)

c) \( \forall x \exists y Q(x, y) \lor (\exists x \forall y P(x, y) \land \neg \exists x \exists y P(x, y)) \)

d) \( \neg (\forall x \exists y P(x, y) \rightarrow \exists x \exists y R(x, y)) \land \forall x \neg \exists y Q(x, y) \)

7. Use the rules in Lemma 9.1 [NS, 1997, page 129] to obtain rules for the following cases.

a) \( \forall x \phi(x) \rightarrow \psi \)

b) \( \exists x \phi(x) \rightarrow \psi \)

c) \( \phi \rightarrow \forall x \psi(x) \)

8. Transform the following sentences into clausal form.

a) \( \neg \exists x ((P(x) \rightarrow P(a)) \land (P(x) \rightarrow P(b))) \)

b) \( \forall y \exists x P(x, y) \)

c) \( \neg \forall y \exists z G(x, y) \)

b) \( \exists x \forall y \exists z (P(x, z) \lor P(z, y) \rightarrow G(x, y)) \)