T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 9 ([NS, 1997], Predicate Logic, Chapters 6 – 7)
April 3–4, and 12, 2007

Solutions to demonstration problems

4. Use semantic tableaux to see whether the following claims holds.

a) $\{\forall x \exists y (P(x) \to Q(y)), \forall x P(x)\} \models \forall x Q(x)$

b) $\{\forall x \forall y (\exists z (R(x,z) \land R(z,y)) \to R(x,y)), R(a,b), R(b,a)\} \models R(a,a)$

Solution.

a)

1. $T(\forall x \exists y (P(x) \to Q(y)))$
2. $T(\forall x P(x))$
3. $F(\forall x Q(x))$
4. $F(Q(c))$ 3. x/c new
5. $T(P(c))$ 2. x/c
6. $T(\exists y (P(c) \to Q(y)))$ 1. x/c
7. $T(P(c) \to Q(d))$ 6. y/d new

8. $F(P(c))$ 7. $\otimes$ 8. $T(Q(d))$ 7.

9. $T(P(d))$ 2. x/d

It seems that the tableaux cannot be finished. We read a counter-example $\mathcal{S}$ from an open branch: domain $U = \{1,2\}$, interpretations for constants $c^{\mathcal{S}} = 1$ and $d^{\mathcal{S}} = 2$, and interpretations for predicates $P^{\mathcal{S}} = \{1,2\}$ and $Q^{\mathcal{S}} = \{2\}$.

Since the *tableau is not finished*, we *need to check the counter-example*. Now, we get $\mathcal{S} \models \forall x \exists y (P(x) \to Q(y))$, $\mathcal{S} \models \forall x P(x)$ and $\mathcal{S} \not\models \forall x Q(x)$ for $\mathcal{S}$.

b)

1. $T(\forall x \forall y (\exists z (R(x,z) \land R(z,y)) \to R(x,y)))$
2. $T(R(a,b))$
3. $T(R(b,a))$
4. $F(R(a,a))$
5. $T(\exists z (R(a,z) \land R(z,a)) \to R(a,a))$, 1. $x/a, y/a$

6. $F(\exists z (R(a,z) \land R(z,a)))$, 5.   7. $T(R(a,a))$, 5. $\otimes$

7. $F(R(a,b) \land R(b,a))$, 6. z/b

$FR(a,b)$ $\otimes$ $FR(b,a)$ $\otimes$

All the branches in the tableau are contradictory and thus the claim holds.

5. We know that

(i) All guilty persons are liars.

(ii) At least one of the accused is also a witness.

(iii) No witness lies.

Use semantic tableaux to prove that all accused are not guilty.

Solution. We choose the following predicates:

$G(x)$ = "x is guilty",
$L(x)$ = "x is liar",
$A(x)$ = "x is accused", and
$W(x)$ = "x is witness".

The sentences are:

(i) $\forall x (G(x) \to L(x))$,

(ii) $\exists x (A(x) \land W(x))$, ja

(iii) $\forall x (W(x) \to \neg L(x))$.

and we want to show that $\neg \forall x (A(x) \to G(x))$. The tableaux proof is as follows.

1. $T(\forall x(G(x) \to L(x)))$
2. $T(\exists x(A(x) \wedge W(x)))$
3. $T(\forall x(W(x) \to \neg L(x)))$
4. $F(\neg \forall x(A(x) \to G(x)))$

5. $T(\forall x(A(x) \to G(x)))$

6. $T(A(a) \wedge W(a))$

7. $T(A(a))$
8. $T(W(a))$

9. $T(A(a) \to G(a))$

10. $F(A(a))$ $\otimes$ | 10. $T(G(a))$

11. $T(W(a) \to \neg L(a))$

12. $F(W(a))$ $\otimes$ | 12. $T(\neg L(a))$

13. $F(L(a))$

14. $T(G(a) \to L(a))$

15. $F(G(a))$ $\otimes$ | 15. $T(L(a))$ $\otimes$

6. We know that:

1) If a brick is on another brick, then it is not on the table.
2) Every brick is either on the table or on another brick.
3) No brick is on a brick which is also on some other brick.

Use semantic tableaux to prove that if a brick is on another brick, the other brick is on the table.

Solution. We use the following predicates:

$T(x,y)$	=	"brick x is on brick y", and
$P(x)$	=	"brick x is on the table".

The set of sentences is:

$$\{\forall x\,(\exists y\, T(x,y) \to \neg P(x)),\ \forall x\,(P(x) \vee \exists y\, T(x,y)),$$
$$\forall x \forall y\,(\exists z\, T(y,z) \to \neg T(x,y))\}$$

and we want to show that $\forall x \forall y\,(T(x,y) \to P(y))$.

Tableau proof:

1. $T(\forall x(\exists y T(x,y) \to \neg P(x)))$
2. $T(\forall x(P(x) \vee \exists y T(x,y)))$
3. $T(\forall x \forall y(\exists z T(y,z) \to \neg T(x,y)))$
4. $F(\forall x \forall y(T(x,y) \to P(y)))$
5. $F(\forall y(T(c,y) \to P(y)))^{4.\ x/c}$ new
6. $F(T(c,d) \to P(d))^{5.\ y/d}$ new
7. $T(T(c,d))^{6.}$
8. $F(P(d))^{6.}$
9. $T(P(d) \vee \exists y T(d,y))$

10. $T(P(d))^{9.}$ $\otimes$ | 10. $T(\exists y T(d,y))^{9.}$

11. $T(T(d,e))^{10.\ y/e}$ new
12. $T(\exists z T(d,z) \to \neg T(c,d))^{3.\ x/c, y/d}$

13. $F(\exists z T(d,z))^{12.}$ | 13. $T(\neg T(c,d))^{12.}$

14. $F(T(d,e))^{13.\ z/e}$ $\otimes$ | 14. $F(T(c,d))^{13.}$ $\otimes$

Note: 1) can be equivalently stated as $\forall x \forall y(T(x,y) \to \neg P(x))$ and 3) as $\forall x \forall y \forall z(T(y,z) \to \neg T(x,y))$. How would the tableau look if you used these sentences?