T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 5 ([Nerode and Shore, 1997], Chapters 4 and 8)
February 20–22, 2007

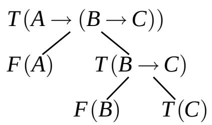
Solutions to demonstration problems

4. Find disjunctive and conjunctive normal forms for the following propositions using (1) the transformation rules and (2) semantic tableaux.

a) $A \to (B \to C)$
Solution. Start by removing implications.

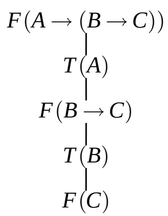
$$\begin{aligned} A \to (B \to C) &\equiv \neg A \vee (\neg B \vee C) \\ &\equiv \neg A \vee \neg B \vee C. \end{aligned}$$

This results in both the conjunctive and the disjunctive normal form for the proposition. When using semantic tableaux for finding the disjunctive normal form for ϕ, one starts with $(T\phi)$.



Now one can reads the disjuncts from the open branches. In this case each of them only contains one literal. Thus we get $\neg A \vee \neg B \vee C$, which is (of course) the same as obtained by applying the transformation rules.

For conjunctive normal form, one starts from $(F\phi)$.



We get $A \wedge B \wedge \neg C$ from the open branch, and this is the disjunctive normal form for the negation of the original proposition. Negating this, we get the conjunctive normal form for the original proposition by applying de Morgan rules.

$$\begin{aligned} A \to (B \to C) &\equiv \neg\neg(A \to (B \to C)) \\ &\equiv \neg(\neg(A \to (B \to C))) \\ &\equiv \neg(A \wedge B \wedge \neg C) \\ &\equiv \neg A \vee \neg B \vee C. \end{aligned}$$

b) $\neg A \leftrightarrow ((A \vee \neg B) \to B)$
Solution. One removes equivalence and implications first, then push negations in front of atomic propositions and finally, apply the distributivity of disjucntion over conjunction.

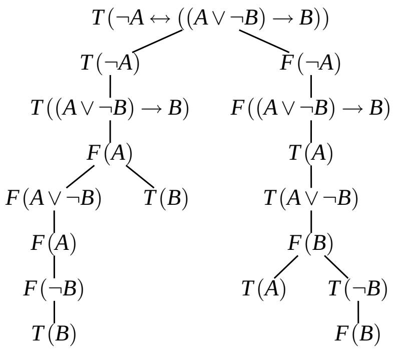
$$\begin{aligned} &\neg A \leftrightarrow ((A \vee \neg B) \to B) \\ &\quad \equiv (\neg A \to ((A \vee \neg B) \to B)) \wedge (((A \vee \neg B) \to B) \to \neg A) && [\leftrightarrow e] \\ &\quad \equiv (A \vee (\neg(A \vee \neg B) \vee B)) \wedge (\neg(\neg(A \vee \neg B) \vee B) \vee \neg A) && [\to e] \\ &\quad \equiv (A \vee ((\neg A \wedge B) \vee B)) \wedge (((A \vee \neg B) \wedge \neg B) \vee \neg A) && [\neg\ s] \\ &\quad \equiv (A \vee ((\neg A \vee B) \wedge (B \vee B))) \wedge ((A \vee \neg B \vee \neg A) \wedge (\neg B \vee \neg A)) && [\wedge\ u] \\ &\quad \equiv (A \vee \neg A \vee B) \wedge (A \vee B) \wedge (A \vee \neg A \vee B) \wedge (\neg A \vee \neg B) && [\wedge\ u] \\ &\quad \equiv (A \vee B) \wedge (\neg A \vee \neg B). \end{aligned}$$

This is the conjunctive normal form. In the last step, we have removed disjunctions of the form $A \vee \neg A \vee B$ because these are always true, that is, $A \vee \neg A \vee B \equiv \top$. Now, to get the disjunctive normal form, we apply the distributivity of conjunction:

$$\begin{aligned} &(A \vee B) \wedge (\neg A \vee \neg B) \\ &\quad \equiv (A \wedge (\neg A \vee \neg B)) \vee (B \wedge (\neg A \vee \neg B)) && [\vee\ u] \\ &\quad \equiv (A \wedge \neg A) \vee (A \wedge \neg B) \vee (\neg A \wedge B) \vee (B \wedge \neg B) && [\vee\ u] \\ &\quad \equiv (A \wedge \neg B) \vee (\neg A \wedge B) \end{aligned}$$

In the last step, we have eliminated multible occurences of same literal in one conjunct and the conjuncts that are always false (containing literal and its complement).

Same with semantic tableaux.



From open braches we get the disjunctive normal form: $(A \wedge \neg B) \vee (\neg A \wedge B)$. Conjunctive normal form can be obtained similarly to item a).

c) $\neg((A \leftrightarrow \neg B) \to C)$
Solution.

$$\begin{aligned} &\neg((A \leftrightarrow \neg B) \to C) \\ &\quad \equiv \neg((A \to \neg B) \wedge (\neg B \to A) \to C) && [\leftrightarrow e] \\ &\quad \equiv \neg(\neg((\neg A \vee \neg B) \wedge (\neg\neg B \vee A)) \vee C) && [\to e] \\ &\quad \equiv (\neg A \vee \neg B) \wedge (A \vee B) \wedge \neg C \ (*) && [\neg\ s] \end{aligned}$$

This is the conjunctive normal form. We continue to obtain the disjunctive normal form.

$$\begin{aligned} (*) &\equiv (\neg A \vee \neg B) \wedge ((B \wedge \neg C) \vee (A \wedge \neg C)) && [\vee\ u] \\ &\equiv (\neg A \wedge ((B \wedge \neg C) \vee (A \wedge \neg C))) \vee \\ &\qquad (\neg B \wedge ((B \wedge \neg C) \vee (A \wedge \neg C))) && [\vee\ u] \\ &\equiv (\neg A \wedge B \wedge \neg C) \vee (\neg A \wedge A \wedge \neg C) \vee \\ &\qquad (\neg B \wedge B \wedge \neg C) \vee (\neg B \wedge A \wedge \neg C) && [\vee\ u] \\ &\equiv (\neg A \wedge B \wedge \neg C) \vee (A \wedge \neg B \wedge \neg C). \end{aligned}$$

d) $P_1 \wedge P_2 \leftrightarrow (P_1 \to P_2) \vee (P_2 \to P_3)$
Solution. One can notice that the term on the right-hand side of the equivalence is valid (check!), and to ease the task we can replace it with $\top$.

$$\begin{aligned} &P_1 \wedge P_2 \leftrightarrow \top \\ &\quad \equiv (P_1 \wedge P_2 \to \top) \wedge (\top \to P_1 \wedge P_2) && [\leftrightarrow e] \\ &\quad \equiv (\neg(P_1 \wedge P_2) \vee \top) \wedge (\neg\top \vee (P_1 \wedge P_2)) && [\to e] \\ &\quad \equiv (\neg P_1 \vee \neg P_2 \vee \top) \wedge (\bot \vee P_1) \wedge (\bot \vee P_2) && [\neg\ s] \\ &\quad \equiv P_1 \wedge P_2. \end{aligned}$$

This is both CNF and DNF.

5. Use semantic tableaux to prove that the rules used to find CNF/DNF of a proposition maintain logical equivalence.

Solution. Use semantic tableuax to proof the validity of $(\alpha \leftrightarrow \beta) \leftrightarrow ((\alpha \to \beta) \wedge (\beta \to \alpha))$, $(\alpha \to \beta) \leftrightarrow (\neg\alpha \vee \beta)$, $\alpha \leftrightarrow \neg\neg\alpha$, etc.

6. Find CNFs for the following propositions both by applying the transformation rules and using semantic tableaux.

a) $(P \wedge \neg P) \vee (Q \wedge \neg Q)$

b) $(P_1 \wedge \neg P_1) \vee \cdots \vee (P_n \wedge \neg P_n)$

Use semantic tableaux to prove that CNF obtained for a) is unsatisfiable.

Solution.

a)

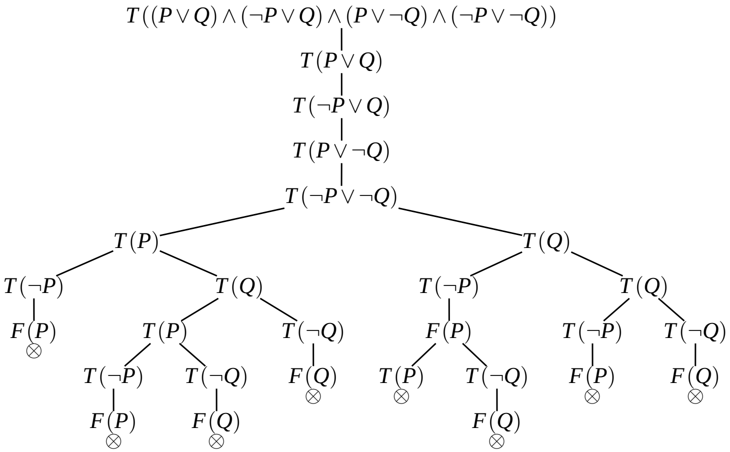
$$\begin{aligned} &(P \wedge \neg P) \vee (Q \wedge \neg Q) \\ &\quad \equiv ((P \wedge \neg P) \vee Q) \wedge ((P \wedge \neg P) \vee \neg Q) \\ &\quad \equiv (P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q) \end{aligned}$$

Semantic tableaux is used similarly to 4. a).

b)

$$\begin{aligned} &(P_1 \wedge \neg P_1) \vee \cdots \vee (P_n \wedge \neg P_n) \\ &\quad \equiv (P_1 \vee \cdots \vee P_n) \wedge (\neg P_1 \vee P_2 \vee \cdots \vee P_n) \wedge \cdots \wedge (\neg P_1 \vee \cdots \vee \neg P_n) \end{aligned}$$

Proposition ϕ is unsatisfiable iff when starting from $(T\phi)$ all branches are contradictory.



7. Find a clause form for $(A\to((A\to A)\to A))\to((A\to(A\to A))\to(A\to A))$

Solution. Remove implications.

$$
\begin{array}{rlcl}
 & (A\to((A\to A)\to A)) & \to & ((A\to(A\to A))\to(A\to A)) \\
\equiv & \neg(A\to((A\to A)\to A)) & \vee & ((A\to(A\to A))\to(A\to A)) \\
\equiv & \neg(\neg A\vee((A\to A)\to A)) & \vee & ((\neg A\vee(A\to A))\to(A\to A)) \\
\equiv & \neg(\neg A\vee(\neg(\neg A\vee A)\vee A)) & \vee & (\neg(\neg A\vee(\neg A\vee A))\vee(\neg A\vee A))
\end{array}
$$

Push negations in front of atomic propositions.

$$
\begin{array}{rlcl}
 & \neg(\neg A\vee(\neg(\neg A\vee A)\vee A)) & \vee & (\neg(\neg A\vee(\neg A\vee A))\vee(\neg A\vee A)) \\
\equiv & (\neg\neg A\wedge\neg(\neg(\neg A\vee A)\vee A)) & \vee & ((\neg\neg A\wedge\neg(\neg A\vee A))\vee(\neg A\vee A)) \\
\equiv & (A\wedge(\neg\neg(\neg A\vee A)\wedge\neg A)) & \vee & ((A\wedge\neg(\neg A\vee A))\vee(\neg A\vee A)) \\
\equiv & (A\wedge((\neg A\vee A)\wedge\neg A)) & \vee & ((A\wedge(A\wedge\neg A))\vee(\neg A\vee A))
\end{array}
$$

Use distributivity rules to push disjuctions inside of conjunctions.

$$
\begin{array}{rlcl}
 & (A\wedge((\neg A\vee A)\wedge\neg A)) & \vee & ((A\wedge(A\wedge\neg A))\vee(\neg A\vee A)) \\
\equiv & (A\wedge((\neg A\vee A)\wedge\neg A)) & \vee & ((A\vee(\neg A\vee A))\wedge((A\wedge\neg A)\vee(\neg A\vee A))) \\
\equiv & (A\wedge((\neg A\vee A)\wedge\neg A)) & \vee & ((A\vee\neg A\vee A)\wedge(A\vee\neg A\vee A)\wedge(\neg A\vee\neg A\vee A))
\end{array}
$$

$$
\begin{array}{rl}
\equiv & (A\vee((A\vee\neg A\vee A)\wedge(A\vee\neg A\vee A)\wedge(\neg A\vee\neg A\vee A))\wedge \\
 & (\neg A\vee A)\wedge\neg A)\vee((A\vee\neg A\vee A)\wedge(A\vee\neg A\vee A)\wedge(\neg A\vee\neg A\vee A)) \\
\equiv & (A\vee A\vee\neg A\vee A)\wedge(A\vee A\vee\neg A\vee A)\wedge(A\vee\neg A\vee\neg A\vee A)\wedge \\
 & (\neg A\vee A\vee A\vee\neg A\vee A)\wedge(\neg A\vee A\vee A\vee\neg A\vee A)\wedge \\
 & (\neg A\vee A\vee\neg A\vee\neg A\vee A)\wedge(\neg A\vee A\vee\neg A\vee A)\wedge(\neg A\vee A\vee\neg A\vee A)\wedge \\
 & (\neg A\vee\neg A\vee\neg A\vee A)
\end{array}
$$

When we eliminate the disjunctions that contain literal and its complement, we notice that all the 9 clauses areeliminated. Thus the resulting set of clauses is empty ($\emptyset$). This should be the case, because the proposition is valid (you can check this using, for example, semantic tableaux).

8. Consider the set of clauses:

$$
\begin{array}{rcl}
S & = & \{\{A_0,A_1\},\{\neg A_0,\neg A_1\},\{A_1,A_2\},\{\neg A_1,\neg A_2\},\ldots, \\
 & & \{A_{n-1},A_n\},\{\neg A_{n-1},\neg A_n\},\{A_n,A_0\},\{\neg A_n,\neg A_0\}\}
\end{array}
$$

Give truth assignment $\mathcal{A}$ such that $\mathcal{A}\models S$.

Solution. Consider the two first clauses of S. We can interpret them as proposition $(A_0\vee A_1)\wedge(\neg A_0\vee\neg A_1)$. This proposition has models $\mathcal{A}_1=\{A_0\}$ and $\mathcal{A}_2=\{A_1\}$, that is, it models the exclusive-or operation (XOR). Thus the set of clauses S is equivalent to proposition

$$(A_0\underline{\vee}A_1)\wedge(A_1\underline{\vee}A_2)\wedge\cdots\wedge(A_n\underline{\vee}A_0).$$

Now, we consider the models of the above propostion for two values of n. When $n=1$ the proposition is $(A_0\underline{\vee}A_1)\wedge(A_1\underline{\vee}A_0)$. If A_0 is true, it implies that A_1 has to be false. Now both conjuncts are satisfied. On the other hand, if A_0 is false, A_1 must be true. The models of S are thus $\{A_0\}$ and $\{A_1\}$.

Now, if $n=2$, the proposition is of the form $(A_0\underline{\vee}A_1)\wedge(A_1\underline{\vee}A_2)\wedge(A_2\underline{\vee}A_0)$. If A_0 is true, then A_1 must be false and furthermore A_2 has to be true. The last XOR demands that A_0 is false if A_2 is true and because of this contradiction there is no model such that A_0 is true. Similar contradiction appears if one assumes that A_0 is false. Thus S has no models for $n=2$.

This can be generalized for all n. If n is odd, S has two models,

$$\{A_0,A_2,\ldots,A_{n-1}\}$$

and

$$\{A_1,A_3,\ldots,A_n\},$$

and if n is even, S has no models (prove the general case!).

9. Horn-clause is a clause that has exactly one positive literal. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be models for a set of Horn-clauses S. Show that also $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2$ is a model of S.

Solution. Assume the opposite, that is, $\mathcal{A} \not\models S$. Then there is a clause $\{A, \neg B_1, \ldots, \neg B_n\}$ in S that is not satisfied. Thus $\{B_1, \ldots, B_n\} \subseteq \mathcal{A}$ (that is, $\mathcal{A} \models B_i$ for all $1 \leq i \leq n$) and $A \notin \mathcal{A}$ (that is, $\mathcal{A} \not\models A$). Based on the definition of intersection $\{B_1, \ldots, B_n\} \subseteq \mathcal{A}_1$ and $\{B_1, \ldots, B_n\} \subseteq \mathcal{A}_2$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ are models of S, then also $A \in \mathcal{A}_1$ and $A \in \mathcal{A}_2$. This implies $A \in \mathcal{A}$ by the definition of intersection, a contradiction. Thus $\mathcal{A} \models S$. $\square$