T-79.3001 Logic in computer science: foundations **Spring 2007**
Exercise 4 ([Nerode and Shore, 1997], Chapters 4 and 7)
February 13–15, 2007

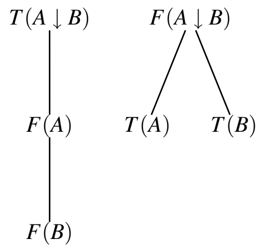
Solutions to demonstration problems

4. Peirce arrow is defined as:

$$A \downarrow B \Leftrightarrow_{def} \neg A \land \neg B.$$

Define semantic tableaux rules for it.

Solution. Based on the definition and the semantic tableaux rules for basic connectives, we get the following rules for Peirce arrow:



5. Use semantic tableux to show that the following propositions are valid.

a) $A \to (B \to B)$,

b) $(A \to B) \land (B \to C) \to (A \to C)$,

c) $(A \to B) \land (A \to C) \to (A \to B \land C)$ ja

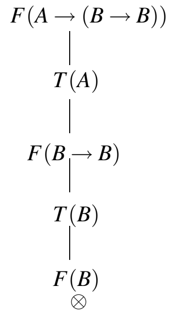
d) $(A \to C) \land (B \to C) \land (A \lor B) \to C$.

We will proceed by constructing semantic tableaux for the negations of the propositions ($E(\phi)$). If all branches close (that is, there are contradictions) then ϕ is valid. If a branch is closed before the tableau is ready, then it is not necessary to continue working on that branch.

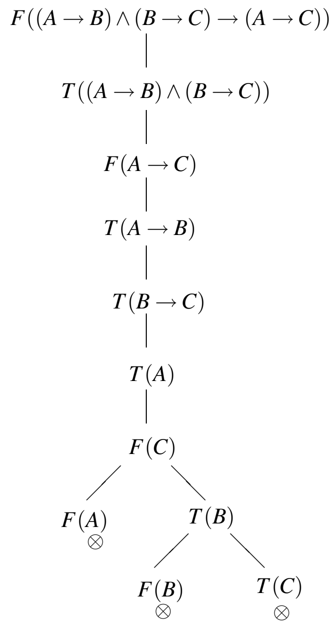
You should notice, that the semantic tableu is actually used to find models for $\neg\phi$. If all branches are contradictionary, then $\neg\phi$ doesn't have a model and its negation is valid.

Solution.

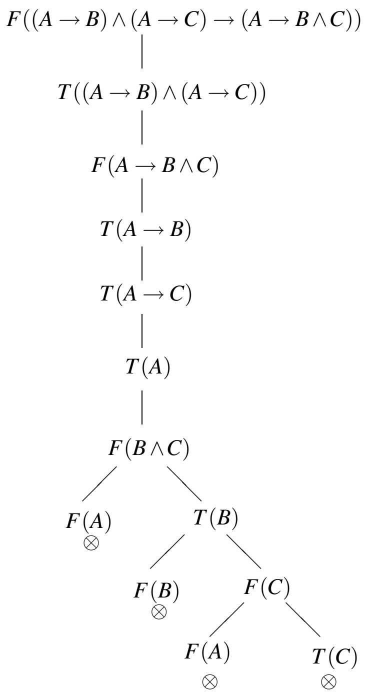
a) $A \to (B \to B)$:



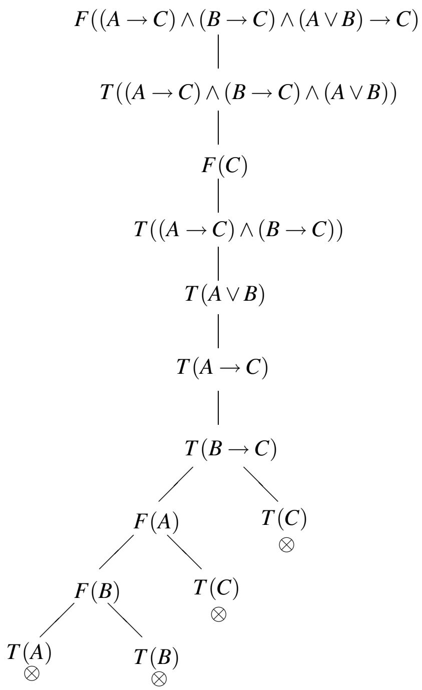
b) $(A \to B) \land (B \to C) \to (A \to C)$:



c) $(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$:



d) $(A \to C) \wedge (B \to C) \wedge (A \vee B) \to C$:



6. Use a semantic tableaux to check whether following claims hold. If not, give a counterexample.

a) $\{B \to A, C \to B, (C \to A) \to D\} \models D$

b) $\{A \to C, A \vee B, \neg D \to \neg B\} \models C \to D$

c) $\models (A \to (B \to C)) \to ((A \to C) \to (A \to B))$

d) $\models (\neg B \to (A \to C)) \to (A \to (B \vee C))$

Solution. When we are checking whether a proposition P is a logical consequence of a set of propositions S we put all node $T(\alpha)$ to the semantic tableaux for all $\alpha \in S$. Next we add $F(P)$ to the tableaux and use inference rules to complete it. If all branches of the tableaux end in a contradiction, we know that P can't be false if all propositions in S are true and so P is a logical consequence. Otherwise, the claim doesn't hold and we can construct a counterexample from an uncontradictionary branch.

a)

- $T(B \to A)$
- $T(C \to B)$
- $T((C \to A) \to D)$
- $F(D)$
  - $F(C \to A)$
    - $T(C)$
    - $F(A)$
      - $F(C)$ $\otimes$
      - $T(B)$
        - $F(B)$ $\otimes$
        - $T(A)$ $\otimes$
  - $T(D)$ $\otimes$

As all brances are contradictory, D is a logical consequence of the set.

b)

- $T(A \to C)$
- $T(A \vee B)$
- $T(\neg D \to \neg B)$
- $F(C \to D)$
- $T(C)$
- $F(D)$
  - $F(\neg D)$
    - $T(D)$ $\otimes$
  - $T(\neg B)$
    - $F(B)$
      - $T(A)$
        - $F(A)$ $\otimes$
        - $T(C)$
      - $T(B)$ $\otimes$

As there is an unclosed branch, $C \to D$ is not logical consequence of the set. We can construct a counter example from the open branch: $\mathcal{A} = \{A, C\}$. Thus it holds $\mathcal{A} \models A \to C$, $\mathcal{A} \models A \vee B$, $\mathcal{A} \models \neg D \to \neg B$, ja $\mathcal{A} \not\models C \to D$ (check!).

c) $\models \phi$ denotes that ϕ is valid. To prove this we construct a semantic tableuax for $F(\phi)$.

- $F((A \to (B \to C)) \to ((A \to C) \to (A \to B)))$
- $T(A \to (B \to C))$
- $F((A \to C) \to (A \to B))$
- $T(A \to C)$
- $F(A \to B)$
- $T(A)$
- $F(B)$
  - $F(A)$ $\otimes$
  - $T(C)$
    - $F(A)$ $\otimes$
    - $T(B \to C)$[6]
      - $F(B)$
      - $T(C)$

Since there is an unclosed brach, the proposition is not valid. A counter example can be constructed from an open branch, for example from the rightmost open branch we get: $\mathcal{A} = \{A, C\}$.

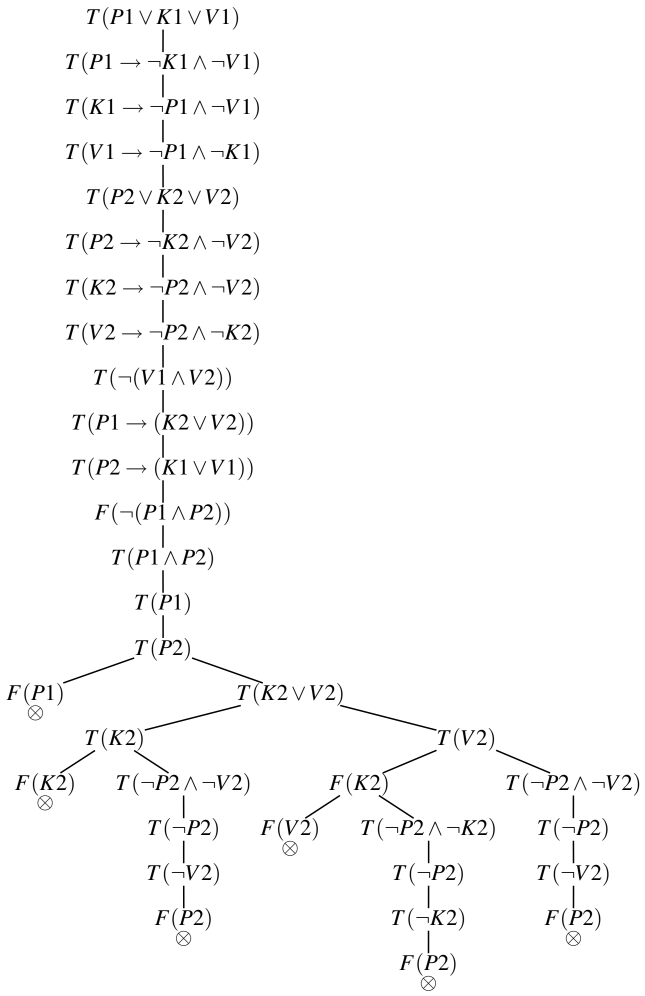
d)

- $F((\neg B \to (A \to C)) \to (A \to B \vee C))$
- $T(\neg B \to (A \to C))$
- $F(A \to B \vee C)$
- $T(A)$
- $F(B \vee C)$
- $F(B)$
- $F(C)$
  - $F(\neg B)$
    - $T(B)$ $\otimes$
  - $T(A \to C)$
    - $F(A)$ $\otimes$
    - $T(C)$ $\otimes$

As all brances are contradictory, the proposition is valid.

7. Recall the specification for two traffic light posts positioned in the intersection of two one-way streets discussed earlier in tutorials. Use semantic tableaux to prove that "the red lights can't be on at the same" is a logical consequence of the set of propositions describing the behaviour of the system.

Solution.



8. Use the proof system by Hilbert to prove the following.

a) $\vdash P \to P$

b) $\{P \to Q, Q \to R\} \vdash P \to R$

c) $\{P, Q \to (P \to R)\} \vdash Q \to R$

Solution.

a)

1.	$(P \to ((P \to P) \to P))$	[A1] $\alpha = P$, $\beta = P \to P$
2.	$((P \to ((P \to P) \to P)) \to$ $((P \to (P \to P)) \to (P \to P)))$	[A2] $\alpha = \gamma = P$, $\beta = P \to P$
3.	$((P \to (P \to P)) \to (P \to P))$	[MP:1,2]
4.	$(P \to (P \to P))$	[A1] $\alpha = P$, $\beta = P$
5.	$(P \to P)$	[MP:3,4]

b)

1.	$(Q \to R)$	[P2]
2.	$((Q \to R) \to (P \to (Q \to R)))$	[A1] $\alpha = Q \to R$, $\beta = P$
3.	$(P \to (Q \to R))$	[MP:1,2]
4.	$((P \to (Q \to R)) \to ((P \to Q) \to (P \to R)))$	[A2] $\alpha = P$, $\beta = Q$, $\gamma = R$
5.	$((P \to Q) \to (P \to R))$	[MP:3,4]
6.	$(P \to Q)$	[P1]
7.	$(P \to R)$	[MP:5,6]

c)

1.	P	[P1]
2.	$(Q \to (P \to R))$	[P2]
3.	$(P \to (Q \to P))$	[A1] $\alpha = P$, $\beta = Q$
4.	$(Q \to P)$	[MP:1,3]
5.	$((Q \to (P \to R)) \to ((Q \to P) \to (Q \to R)))$	[A2] $\alpha = Q$, $\beta = P$, $\gamma = R$
6.	$((Q \to P) \to (Q \to R))$	[MP:2,5]
7.	$(Q \to R)$	[MP:4,6]